# PROOF OF FORMULA 4.253.2

$$\int_0^1 \frac{x^{p-1}\,\ln x\,dx}{(1-x)^{p+1}} = -\frac{\pi}{p\,\sin \pi p}$$

Differentiating the identity

$$B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx,$$

with respect to $a$ yields

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,\ln x\,dx = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\left(\psi(a)-\psi(a+b)\right).$$

Write the integral as

$$\begin{aligned}\int_0^1 \frac{x^{p-1}\,\ln x\,dx}{(1-x)^{p+1}} &= \lim_{\epsilon\to 0}\int_0^1 x^{p-1+\epsilon}(1-x)^{-p-1}\,\ln x\,dx \\ &= \lim_{\epsilon\to 0}\frac{\Gamma(p+\epsilon)\Gamma(-p)}{\Gamma(\epsilon)}\left(\psi(p)-\psi(\epsilon)\right).\end{aligned}$$

Now use $\Gamma(\epsilon) = 1/\epsilon - \gamma + o(1)$ to obtain

$$\frac{\psi(p)-\psi(\epsilon)}{\Gamma(\epsilon)} = \frac{\Gamma(\epsilon)\psi(p)-\Gamma'(\epsilon)}{\Gamma^2(\epsilon)} \to 1 \text{ as } \epsilon \to 0.$$

Therefore

$$\int_0^1 \frac{x^{p-1}\,\ln x\,dx}{(1-x)^{p+1}} = \Gamma(p)\Gamma(-p).$$

The result follows from $\Gamma(1-p) = -p\Gamma(p)$ and $\Gamma(p)\Gamma(1-p) = \pi/\sin \pi p$.