PROOF OF FORMULA 4.253.6

$$\int_0^\infty \frac{\ln x \, dx}{(a+x)^{\mu+1}} = \frac{1}{\mu a^\mu} \left(\ln a - \gamma - \psi(\mu) \right)$$

The change of variable $x = at$ produces

$$\int_0^\infty \frac{\ln x \, dx}{(a+x)^{\mu+1}} = \frac{1}{\mu a^\mu} \frac{\ln a}{a^\mu} \int_0^\infty \frac{dt}{(1+t)^{\mu+1}} + \frac{1}{a^\mu} \int_0^\infty \frac{\ln t \, dt}{(1+t)^{\mu+1}}.$$

The first integral is elementary with value $1/\mu$. The second one comes from differentiating the relation

$$B(x, b-x) = \frac{\Gamma(x)\Gamma(b-x)}{\Gamma(b)} = \int_0^\infty \frac{t^{x-1} \, dt}{(1+t)^b}$$

with respect to b to poroduce

$$\int_0^\infty \frac{\ln t \, dt}{(1+t)^{\mu+1}} = -\frac{1}{\mu} \left[\gamma + \psi(\mu) \right].$$

This gives the result.