PROOF OF FORMULA 4.272.12

$$\int_0^1 \left(\ln \frac{1}{x} \right)^{\mu-1} \frac{x^{\nu-1}\,dx}{1-x^2} = \frac{\Gamma(\mu)}{2^\mu} \zeta \left(\mu, \frac{\nu}{2} \right)$$

The change of variables $t = \ln \frac{1}{x}$ yields

$$\int_0^1 \left(\ln \frac{1}{x} \right)^{\mu-1} \frac{x^{\nu-1}\,dx}{1-x^2} = \int_0^\infty \frac{t^{\mu-1} e^{-\nu t}}{1-e^{-2t}}\,dt.$$

Expand the integrand as a geometric series to obtain

$$\begin{aligned}
\int_0^1 \left(\ln \frac{1}{x} \right)^{\mu-1} \frac{x^{\nu-1}\,dx}{1-x^2} &= \sum_{k=0}^\infty \int_0^\infty t^{\mu-1} e^{-(\nu+2k)t} dt \\
&= \Gamma(\mu) \sum_{k=0}^\infty \frac{1}{(\nu+2k)^\mu} \\
&= \frac{\Gamma(\mu)}{2^\mu} \sum_{k=0}^\infty \frac{1}{(k+\nu/2)^\mu}.
\end{aligned}$$

This is the result.