PROOF OF FORMULA 4.272.19

$$\begin{aligned}\int_0^1 \left(\ln \frac{1}{x}\right)^{2n-1} \frac{x^p - x^{-p}}{1-x^q} x^{q-1}\, dx &= \frac{1}{p^{2n}} \sum_{k=n}^{\infty} \left(\frac{2\pi p}{q}\right)^k \frac{|B_{2k}|}{2k(2k-2n)!} \\ &= \frac{\Gamma(2n)}{q^{2n}} \left[\zeta\left(2n, \frac{p}{q}\right) - \zeta\left(2n, -\frac{p}{q}\right)\right]\end{aligned}$$

Proof. Expanding the term $1/(1-x^q)$ as a geometric series yields

$$I = \sum_{k=0}^{\infty} \int_0^1 \left(\ln \frac{1}{x}\right)^{2n-1} x^{p-1+q(k+1)}\, dx - \sum_{k=0}^{\infty} \int_0^1 \left(\ln \frac{1}{x}\right)^{2n-1} x^{-p-1+q(k+1)}\, dx.$$

The result now follows from the integral

$$\int_0^1 x^a \left(\ln \frac{1}{x}\right)^b dx = \frac{\Gamma(1+b)}{(1+a)^{1+b}}$$

and the definition

$$\zeta(a,s) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}.$$

□