PROOF OF FORMULA 4.291.17

$$\int_0^\infty \frac{\ln(a+x)\,dx}{(b+x)^2} = \frac{a\ln a - b\ln b}{b(a-b)}$$

The change of variables $x = bt$ shows that the entry is equivalent to

$$I(a) := \int_0^\infty \frac{\ln(a+bt)\,dx}{(1+t)^2} = \frac{a\ln a - b\ln b}{a-b}.$$

Differentiating with respect to the parameter a it follows that

$$I'(a) = \int_0^\infty \frac{dt}{(a+bt)\,(1+t)^2}.$$

This integral is evaluated by partial fractions using the decomposition

$$\frac{1}{(a+bt)(1+t)^2} = \frac{1}{(a-b)(1+t)^2} - \frac{b}{(a-b)^2(1+t)} + \frac{b^2}{(a-b)^2(a+bt)},$$

to obtain

$$I'(a) = \frac{1}{a-b} - \frac{b}{(a-b)^2}\ln\frac{a}{b}.$$

Integration yields

$$I(a) = \ln a + \frac{b}{a-b}\ln\frac{a}{b} + C.$$

The fact that the constant of integration C vanishes comes from $I(0) = \ln b$. This gives the result.