PROOF OF FORMULA 4.291.18

$$\int_0^a \frac{\ln(1+ax)}{1+x^2}\,dx = \frac{1}{2}\tan^{-1} a \, \ln(1+a^2)$$

The change of variables $x = \tan t$ gives

$$\begin{aligned} \int_0^a \frac{\ln(1+ax)}{1+x^2}\,dx &= \int_0^u \ln(1+a\tan t)\,dt \\ &= \int_0^u \ln(\cos t + a\sin t)\,dt - \int_0^u \ln\cos t\,dt, \end{aligned}$$

where $u = \tan^{-1} a$.

Now write

$$\cos t + a\sin t = \sqrt{1+a^2}\,\sin(t+v)$$

where $v = \frac{\pi}{2} - u$. This gives

$$\begin{aligned} \int_0^a \frac{\ln(1+ax)}{1+x^2}\,dx &= \int_0^u \ln(1+a\tan t)\,dt \\ &= \int_0^u \ln(\cos t + a\sin t)\,dt - \int_0^u \ln\cos t\,dt \\ &= \frac{1}{2}\ln(1+a^2)\tan^{-1} a + \int_0^u \ln\sin(t+v)\,dt - \int_0^u \ln\cos t\,dt \end{aligned}$$

and the result follows from the identity

$$\int_0^u \ln\sin(t+v)\,dt = \int_v^{\pi/2} \ln\sin t\,dt = \int_0^{\pi/2-v} \ln\cos t\,dt = \int_0^u \ln\cos t\,dt.$$