PROOF OF FORMULA 4.355.4

$$\int_0^\infty (2\mu x^2 - 2n - 1) x^{2n} e^{-\mu x^2} \ln x \, dx = \frac{(2n-1)!!}{2(2\mu)^n} \sqrt{\frac{\pi}{\mu}}$$

In the proof of entry 4.355.1 the formula

$$\int_0^\infty x^m e^{-sx^b} \ln x \, dx = \frac{\Gamma(r)}{b^2 s^r} \left[\psi(r) - \ln s \right]$$

was established. Here $r = (m+1)/b$. Using this in the current problem, the integral

$$\int_0^\infty (2\mu x^2 - 2n - 1) x^{2n} e^{-\mu x^2} \ln x \, dx$$

is reduced to

$$2\mu \int_0^\infty x^{2n+2} e^{-\mu x^2} \ln x \, dx - (2n+1)\mu \int_0^\infty x^{2n} e^{-\mu x^2} \ln x \, dx$$

$$\frac{2\mu\Gamma(n+3/2)}{4\mu^{n+3/2}} \left[\psi\left(n + \frac{3}{2}\right) - \ln \mu \right] - \frac{(2n+1)\mu\Gamma(n+1/2)}{4\mu^{n+1/2}} \left[\psi\left(n + \frac{1}{2}\right) - \ln \mu \right].$$

This reduces to the form stated here using

$$\Gamma\left(n + \tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n} (2n-1)!!$$

given in entry 8.338.1.