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The integrals in Gradshteyn and Ryzhik. Part 24: Polylogarithm functions

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Abstract. The table of Gradshteyn and Ryzhik contains some integrals that can be evaluated using the polylogarithm function. A small selection of examples is discussed.

1. Introduction

The table of integrals [**2**] contains many entries that are expressible in terms of the *polylogarithm function*

$$\mathrm{Li}_s(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^s}. \tag{1.1}$$

In this paper we describe the evaluation of some of them. The series (1.1) converges for $|z| < 1$ and $\mathrm{Re}\, s > 1$. The integral representation

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}\, dx}{e^x - z} \tag{1.2}$$

provides an analytic extension to $\mathbb{C}$. Here $\Gamma(s)$ is the classical *gamma function* defined by

$$\Gamma(s) := \int_0^\infty x^{s-1} e^{-x}\, dx. \tag{1.3}$$

The polylogarithm function is a generalization of the *Riemann zeta function*

$$\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s} = \mathrm{Li}_s(1). \tag{1.4}$$

A second special value is given by

$$\mathrm{Li}_s(-1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} = -(1 - 2^{1-s})\zeta(s), \tag{1.5}$$

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the last equality being obtained by splitting the sum according to the parity of the summation index.

The first result is an identity between an integral and a series coming from the evaluation of the polylogarithm at two values on the unit circle. Many of the entries presented here are special cases. This is a classical result, the proof is presented here in order to keep the paper as self-contained as possible.

THEOREM 1.1. *Let* $\nu \in \mathbb{C}$ *with* $\text{Re}\,\nu > 0$ *and* $0 < t < \pi$. *Then*

$$\int_0^\infty \frac{x^{\nu-1}\,dx}{\cosh x - \cos t} = \frac{2\Gamma(\nu)}{\sin t} \sum_{k=1}^\infty \frac{\sin kt}{k^\nu}. \tag{1.6}$$

PROOF. The integral representation (1.2) gives

$$\begin{aligned} i\left[\text{Li}_s(e^{-it}) - \text{Li}_s(e^{it})\right] &= \frac{i}{\Gamma(s)} \int_0^\infty x^{s-1} \left[\frac{1}{e^{x+it}-1} - \frac{1}{e^{x-it}-1} \right] dx \\ &= \frac{\sin t}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}\,dx}{\cosh x - \cos t}. \end{aligned}$$

The series representation (1.1) gives

$$\begin{aligned} i\left[\text{Li}_s(e^{-it}) - \text{Li}_s(e^{it})\right] &= 2\sum_{k=1}^\infty \frac{e^{ikt} - e^{-ikt}}{2ik^s} \\ &= 2\sum_{k=1}^\infty \frac{\sin kt}{k^s}. \end{aligned}$$

This proves the result. □

COROLLARY 1.1. *Let* $\nu \in \mathbb{C}$ *with* $\text{Re}\,\nu > 0$ *and* $0 < t < \pi$. *Then*

$$\int_0^\infty \frac{x^{\nu-1}\,dx}{\cosh x + \cos t} = \frac{2\Gamma(\nu)}{\sin t} \sum_{k=1}^\infty (-1)^{k-1} \frac{\sin kt}{k^\nu}. \tag{1.7}$$

PROOF. Replace t by $\pi - t$ in the statement of Theorem 1.1. □

This corollary appears as entry **3.531.7** in [**2**].

REMARK 1.1. In the special case $\nu = 2$, the series in the corollary appears in the expansion of the Lobachevsky function

$$L(t) := -\int_0^t \ln\cos s\,ds = t\;\ln 2 - \frac{1}{2}\sum_{k=1}^\infty (-1)^{k-1}\frac{\sin 2kt}{k^2}, \quad 0 < t < \frac{\pi}{2}. \tag{1.8}$$

This special case of the corollary can be stated as

$$\int_0^\infty \frac{x\,dx}{\cosh x + \cos 2t} = \frac{4\left(t\;\ln 2 - L(t)\right)}{\sin 2t}, \quad 0 < t < \frac{\pi}{2}. \tag{1.9}$$

This is entry **3.531.2** of [**2**]. Observe that this is written as

$$\int_0^\infty \frac{x\,dx}{\cosh 2x + \cos 2t} = \frac{t\;\ln 2 - L(t)}{\sin 2t}, \quad 0 < t < \frac{\pi}{2}. \tag{1.10}$$

The fact that this is the only entry in Section 3.531 with $\cosh 2x$ instead of $\cosh x$ can lead to confusion.

2. Some examples from the table by Gradshteyn and Ryzhik

This section presents the evaluation of some entries from the table [**2**] by making specific choices for the parameters ν and t in Theorem 1.1 and Corollary 1.1. Naturally a closed-form for the integral is obtained in those cases for which the series can be evaluated.

Example 2.1. Take $\nu = 2$ and $t = \pi/3$. Theorem 1.1 gives

$$\begin{aligned}\int_0^\infty \frac{x\,dx}{\cosh x - \frac{1}{2}} &= \frac{2\Gamma(2)}{\sin \pi/3} \sum_{k=1}^\infty \frac{\sin(k\pi/3)}{k^2} \\ &= \frac{4}{\sqrt{3}} \sum_{k=1}^\infty \frac{\sin(k\pi/3)}{k^2}.\end{aligned}$$

The function $\sin(\pi k/3)$ is periodic, with period 6, and repeating values $\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0$. Therefore

$$\sum_{k=1}^\infty \frac{\sin\left(\frac{\pi k}{3}\right)}{k^2} = \frac{\sqrt{3}}{2}\left(\sum_{k=0}^\infty \frac{1}{(6k+1)^2} + \sum_{k=0}^\infty \frac{1}{(6k+2)^2} - \sum_{k=0}^\infty \frac{1}{(6k+4)^2} - \sum_{k=0}^\infty \frac{1}{(6k+5)^2} \right).$$

To evaluate this series, recall the series representation of the *digamma* function $\psi(x) = \Gamma'(x)/\Gamma(x)$, given by

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{k=1}^\infty \frac{x}{k(x+k)}, \quad \text{for } x > 0. \tag{2.1}$$

Differentiation yields

$$\psi'(x) = \sum_{k=0}^\infty \frac{1}{(x+k)^2}, \quad \text{for } x > 0, \tag{2.2}$$

and we obtain

$$\sum_{k=0}^\infty \frac{1}{(6k+j)^2} = \frac{1}{36} \sum_{k=0}^\infty \frac{1}{(k+\frac{j}{6})^2} = \frac{1}{36}\psi'\left(\frac{j}{6}\right).$$

This provides the expression

$$\sum_{k=1}^\infty \frac{\sin\left(\frac{\pi k}{3}\right)}{k^2} = \frac{\sqrt{3}}{72}\left(\psi'\left(\frac{1}{6}\right) + \psi'\left(\frac{2}{6}\right) - \psi'\left(\frac{4}{6}\right) - \psi'\left(\frac{5}{6}\right)\right). \tag{2.3}$$

The identities

$$\psi(1-x) = \psi(x) + \pi \cot \pi x, \quad \text{for } 0 < x < 1, \tag{2.4}$$

and

$$\psi(2x) = \frac{1}{2}\left(\psi(x) + \psi(x+\tfrac{1}{2})\right) + \ln 2, \tag{2.5}$$

produce

$$\psi'\left(\frac{1}{6}\right) = 5\psi'\left(\frac{1}{3}\right) - \frac{4\pi^2}{3},\ \psi'\left(\frac{2}{3}\right) = -\psi'\left(\frac{1}{3}\right) + \frac{4\pi^2}{3},\ \psi'\left(\frac{5}{6}\right) = -5\psi'\left(\frac{1}{3}\right) + \frac{16\pi^2}{3}.$$

It follows that

$$\int_0^\infty \frac{x\,dx}{\cosh x - \frac{1}{2}} = \frac{2}{3}\psi'\left(\frac{1}{3}\right) - \frac{4\pi^2}{9}. \tag{2.6}$$

This example appears as entry **3.531.1**. The value stated there is given in terms of the Lobachevsky function using (1.9):

$$\int_0^\infty \frac{x\,dx}{\cosh x - \frac{1}{2}} = \frac{8}{\sqrt{3}}\left(\frac{\pi}{3}\ln 2 - L\left(\frac{\pi}{3}\right)\right). \tag{2.7}$$

Comparing these two evaluations gives

$$L\left(\frac{\pi}{3}\right) = -\frac{1}{4\sqrt{3}}\psi'\left(\frac{1}{3}\right) + \frac{\pi^2}{6\sqrt{3}} + \frac{\pi}{3}\ln 2. \tag{2.8}$$

This example also appears as entry **3.418.1** in the form

$$\int_0^\infty \frac{x\,dx}{e^x + e^{-x} - 1} = \frac{1}{3}\left[\psi'\left(\frac{1}{3}\right) - \frac{2\pi^2}{3}\right]. \tag{2.9}$$

EXAMPLE 2.2. Entry **3.514.1** in [**2**] is

$$\int_0^\infty \frac{dx}{\cosh ax + \cos t} = \frac{t}{a\,\sin t}, \qquad \text{for } 0 < t < \pi,\ a > 0. \tag{2.10}$$

The case of arbitrary $a > 0$ is equivalent to the special case $a = 1$. This follows from the change of variables $ax \mapsto x$. The integral

$$\int_0^\infty \frac{dx}{\cosh x + \cos t} = \frac{t}{\sin t}, \qquad \text{for } 0 < t < \pi, \tag{2.11}$$

is now evaluated by elementary methods.

The next sequence of identities gives the result:

$$\begin{aligned}
\int_0^\infty \frac{dx}{\cosh x + \cos t} &= 2\int_0^\infty \frac{e^x\,dx}{e^{2x} + 2e^x\cos t + 1} \\
&= 2\int_1^\infty \frac{dr}{r^2 + 2r\cos t + 1} \\
&= 2\int_{1+\cos t}^\infty \frac{du}{u^2 + \sin^2 t} \\
&= \frac{2}{\sin t}\int_{\cot(t/2)}^\infty \frac{dv}{v^2+1} \\
&= \frac{t}{\sin t}.
\end{aligned}$$

EXAMPLE 2.3. The exponential generating function for the Bernoulli polynomials $B_n(x)$ is $te^{xt}/(e^t-1)$, so for real x and t with $0<|t|<2\pi$,

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}. \tag{2.12}$$

For $n=2m+1$ an odd integer, these polynomials have a Fourier sine series given by

$$\frac{2^{2m}\pi^{2m+1}(-1)^m}{(2m+1)!}B_{2m+1}\left(\frac{t}{2\pi}+\frac{1}{2}\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{2m+1}}\sin kt, \text{ for } |t|<\pi. \tag{2.13}$$

For example, $n=3$ gives

$$\frac{t(\pi^2-t^2)}{12} = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin kt}{k^3}, \text{ for } |t|<\pi, \tag{2.14}$$

and $n=5$ gives

$$\frac{t(\pi^2-t^2)(7\pi^2-3t^2)}{720} = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin kt}{k^5}, \text{ for } |t|<\pi. \tag{2.15}$$

These representations and Corollary 1.1 give the evaluations

$$\int_0^\infty \frac{x^2\,dx}{\cosh x+\cos t} = \frac{t(\pi^2-t^2)}{3\sin t}, \quad \text{for } 0<t<\pi, \tag{2.16}$$

and

$$\int_0^\infty \frac{x^4\,dx}{\cosh x+\cos t} = \frac{t(\pi^2-t^2)(7\pi^2-3t^2)}{15\sin t}, \quad \text{for } 0<t<\pi. \tag{2.17}$$

These integrals appear as entries **3.531.3** and **3.531.4**, respectively. The Fourier sine series

$$\frac{t}{2} = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin kt}{k}, \text{ for } |t|<\pi, \tag{2.18}$$

shows that the evaluation given in Example 2.2 is also part of this family.

EXAMPLE 2.4. The limiting case $t\to 0$ in Corollary 1.1 gives, for $\nu\neq 2$, the evaluation

$$\int_0^\infty \frac{x^{\nu-1}\,dx}{\cosh x+1} = 2(1-2^{2-\nu})\Gamma(\nu)\zeta(\nu-1). \tag{2.19}$$

The proof uses the elementary limit $\sin kt/\sin t\to k$ as $t\to 0$ and (1.5). The identity (2.19) is part of entry **3.531.6**. An alternative direct proof is presented next.

The integral representation

$$\int_0^\infty \frac{x^{s-1}\,dx}{e^{px}+1} = \frac{(1-2^{1-s})}{p^s}\Gamma(s)\zeta(s) \tag{2.20}$$

appears as entry **9.513.1** in [**2**] and it is established in [**3**] and in [**1**].

Differentiating with respect to p gives

$$\int_0^\infty \frac{x^s e^{px}\,dx}{(e^{px}+1)^2} = \frac{s(1-2^{1-s})}{p^{1+s}}\Gamma(s)\zeta(s) \tag{2.21}$$

and $p = 1/2$ produces

$$\int_0^\infty \frac{x^s\,dx}{e^{x/2} + e^{-x/2} + 2} = 2^{s+1}(1 - 2^{1-s})\Gamma(s+1)\zeta(s). \tag{2.22}$$

The change of variables $u = x/2$ and $\nu = s+1$ give the result.

The limiting case $\nu \to 2$

$$\int_0^\infty \frac{x\,dx}{\cosh x + 1} = 2\ln 2 \tag{2.23}$$

that is also part of entry **3.531.6**, appears from the limiting behavior

$$\zeta(s) = \frac{1}{s-1} + F(s) \tag{2.24}$$

where $F(s)$ is an entire function.

Example 2.5. Let $t = 2\pi a$ in Theorem 1.1 and take $\nu = 2m+1$ with $m \in \mathbb{N}$ to obtain

$$\int_0^\infty \frac{x^{2m}\,dx}{\cosh x - \cos 2\pi a} = \frac{2(2m)!}{\sin 2\pi a} \sum_{k=1}^\infty \frac{\sin 2\pi k a}{k^{2m+1}}. \tag{2.25}$$

This is entry **3.531.5** in [**2**]. The hypotheses of the theorem restrict a to $0 < a < 1/2$, but the symmetry about $a = 1/2$ implies that (2.25) also holds for $1/2 < a < 1$.

In the special case $a = \frac{1}{2}$, replacing $\sin 2\pi k a / \sin 2\pi a$ by its limiting value, produces

$$\int_0^\infty \frac{x^{2m}\,dx}{\cosh x + 1} = 2(1 - 2^{1-2m})(2m)!\zeta(2m), \tag{2.26}$$

in agreement with (2.19). For positive integer m, the relation

$$\zeta(2m) = \frac{2^{2m-1}\pi^{2m}|B_{2m}|}{(2m)!} \tag{2.27}$$

expresses the integral in (2.26) in terms of the Bernoulli numbers B_{2m} as

$$\int_0^\infty \frac{x^{2m}\,dx}{\cosh x + 1} = 2(2^{2m-1} - 1)\pi^{2m}|B_{2m}|. \tag{2.28}$$

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