

A property of Euler's elastic curve

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Abstract

Legendre's formula for the classical elliptic integrals is discussed. We present a generalization of Euler's proof of the lemniscatic case of modulus $k = \sqrt{-1}$ to the curves

$$f_n(x) := \int_0^x \frac{t^n dt}{\sqrt{1-t^{2n}}}.$$

1. INTRODUCTION

During the first two decades of the last century, Legendre developed the theory of elliptic integrals. His work [5] appeared in 1811 and his monumental treatise [6] in 1825. Shortly after that, Abel published his work [1] on the inversion of elliptic integrals and on the properties of the elliptic functions defined by this procedure. One of Legendre's most elegant formulae appears on [5] page 61. This is his famous

relation:

$$\begin{aligned} & \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \times \int_0^1 \sqrt{\frac{1-(k')^2x^2}{1-x^2}} dx + \\ & \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-(k')^2x^2)}} \times \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}} dx - \\ & \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \times \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-(k')^2x^2)}} = \frac{\pi}{2}. \end{aligned} \quad (1.1)$$

The terms in (1.1) are the classical elliptic integrals that made their debut in the calculation of the length of the ellipse and the lemniscate. The reader is referred to [7] for details on this topic and to [2] for the history of Legendre's relation (1.1).

The lemniscatic integral (1.3), below) appears in the calculation of the arclength of the lemniscate of equation $(x^2+y^2)^2 = a^2(x^2-y^2)$. Siegel [8] makes this example his starting point in his book on abelian functions. The parametrization of the lemniscate

$$x = \sqrt{\frac{r^2+r^4}{2}} \quad \text{and} \quad y = \sqrt{\frac{r^2-r^4}{2}}, \quad (1.2)$$

with $r = \sqrt{x^2+y^2}$, yields the expression

$$L = \int_0^1 \frac{dx}{\sqrt{1-x^4}} \quad (1.3)$$

for the total arclength. This lemniscatic integral was studied by Euler in [4] and is the special case $k = \sqrt{-1}$ of the *elliptic integral of the first kind*

$$K(k) := \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

later studied by Legendre in [6]. In this case (1.1) becomes

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} = \frac{\pi}{4}. \quad (1.4)$$

In this paper we describe Euler's method to prove (1.4) and establish a generalization that deals with the elastic curve

$$f_n(x) := \int_0^x \frac{t^n dt}{\sqrt{1-t^{2n}}}$$

for which we prove that

$$R_n \times L_n = \frac{\pi}{2n}$$

where $R_n = f_n(1)$ is the so-called *main radius*, and L_n is the length of the curve from $x = 0$ to $x = 1$. The special case $n = 2$ yields Euler's result.

Section 2 recalls a standard proof of (1.1) based on the fact that the Legendre integrals satisfy a differential equation. Section 3 describes Euler's original proof, its generalization and discusses the issue of convergence, a fact that Euler was happy to

ignore. Although Euler did not explicitly address the issue of convergence in [3], his familiarity with Stirling's formula dates from at least 1736.

2. LEGENDRE'S PROOF

The first proof of Legendre's relation (1.1) is based on a differential equation satisfied by the elliptic integrals

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad \text{and} \quad E(k) = \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}} dx.$$

Among the many identities satisfied by these functions we employ an expression for their derivatives.

Proposition 2.1. *The functions $K(k)$ and $E(k)$ satisfy*

$$\begin{aligned} k(k')^2 \frac{dK}{dk} &= E - (k')^2 K \\ k \frac{dE}{dk} &= E - K, \end{aligned} \tag{2.1}$$

where $k' = \sqrt{1-k^2}$ is the conjugate modulus.

Proof. This follows directly from the definitions. □

Proposition 2.2. *Let $K'(k) = K(k')$ and $E'(k) = E(k')$. Then the function $KE' + EK' - KK'$ is constant.*

Proof. Employ Proposition 2.1 to check that the derivative is identically 0. □

Legendre then evaluates the constant at the modulus $k = \frac{1}{2}\sqrt{2-\sqrt{3}}$ and its complement $k' = \frac{1}{2}\sqrt{2+\sqrt{3}}$. In this paper we complete Legendre's proof by using the modulus $k = \sqrt{-1}$. This is explained in the next section.

3. EULER'S DIRECT PROOF

In [3] Euler developed his theory of infinite products and used it in [4] to prove the relation

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} = \frac{\pi}{4}. \tag{3.1}$$

In this paper we generalize Euler's method and prove the following result.

Theorem 3.1. *The generalized elastic curve*

$$f_n(x) := \int_0^x \frac{t^n dt}{\sqrt{1-t^{2n}}} \tag{3.2}$$

satisfies

$$R_n \times L_n = \frac{\pi}{2n},$$

R_n is the main radius, the value $f_n(1)$, and L_n is the length of the curve from $x = 0$ to $x = 1$.

Proof. We have

$$R_n = \int_0^1 \frac{t^n dt}{\sqrt{1-t^{2n}}}$$

and

$$L_n = \int_0^1 \frac{dt}{\sqrt{1-t^{2n}}}.$$

Integrate the relation

$$d \left(t^k \sqrt{1-t^{2n}} \right) = \frac{kt^{k-1} dt - (k+n)t^{2n+k-1} dt}{\sqrt{1-t^{2n}}}$$

from 0 to 1 to produce the recursive formula

$$\int_0^1 \frac{t^{k-1} dt}{\sqrt{1-t^{2n}}} = \frac{k+n}{k} \int_0^1 \frac{t^{2n+k-1} dt}{\sqrt{1-t^{2n}}}. \quad (3.3)$$

The value $k = n + 1$ in (3.3) yields

$$R_n = \frac{2n+1}{n+1} \int_0^1 \frac{t^{3n} dt}{\sqrt{1-t^{2n}}}. \quad (3.4)$$

Then the value $k = 3n + 1$ produces

$$\int_0^1 \frac{t^{3n} dt}{\sqrt{1-t^{2n}}} = \frac{4n+1}{3n+1} \int_0^1 \frac{t^{5n} dt}{\sqrt{1-t^{2n}}}$$

so (3.4) produces

$$R_n = \frac{2n+1}{n+1} \times \frac{4n+1}{3n+1} \int_0^1 \frac{t^{5n} dt}{\sqrt{1-t^{2n}}}.$$

Iterating (3.3) we obtain, after m steps,

$$R_n = \prod_{j=1}^m \frac{2jn+1}{(2j-1)n+1} \times \int_0^1 \frac{t^{(2m+1)n} dt}{\sqrt{1-t^{2n}}}. \quad (3.5)$$

The next step is to justify the passage to the limit in (3.5) as $m \rightarrow \infty$, with n fixed. Observe that the left hand side is *independent* of m , so it remains R_n after $m \rightarrow \infty$. The difficulty in passing to the limit is that the product in (3.5) diverges. The general term p_j satisfies

$$1 - p_j = \frac{-n}{(2j-1)n+1}$$

and the divergence of the product follows from that of the harmonic series. The divergence is cured by introducing scaling factors both in the integral and the product. The proof is omitted in Eulerian fashion.

Proposition 3.2. *The functions*

$$\frac{1}{2m+1} \int_0^1 \frac{t^{(2m+1)n}}{\sqrt{1-t^{2n}}} dt \text{ and } (2m+1) \times \prod_{j=1}^m \frac{2jn+1}{(2j-1)n+1}$$

have non-zero limits as $m \rightarrow \infty$.

Therefore from (3.5) we obtain

$$R_n = \lim_{m \rightarrow \infty} \prod_{j=1}^{2m} (jn+1)^{(-1)^j} \times \int_0^1 \frac{t^{(2m+1)n}}{\sqrt{1-t^{2n}}} dt$$

where we have employed

$$\prod_{j=1}^m \frac{2jn+1}{(2j-1)n+1} = \prod_{j=1}^{2m} (jn+1)^{(-1)^j}$$

in order to simplify the notation. A similar argument shows that

$$\begin{aligned} L_n &= \prod_{j=1}^m \frac{(2j-1)n+1}{2(j-1)n+1} \int_0^1 \frac{t^{2mn}}{\sqrt{1-t^{2n}}} dt \\ &= \lim_{m \rightarrow \infty} \prod_{j=1}^{2m} (jn+1)^{(-1)^{j+1}} \int_0^1 \frac{t^{2mn}}{\sqrt{1-t^{2n}}} dt \end{aligned} \quad (3.6)$$

The final step is to introduce the auxiliary quantities

$$A_n := \int_0^1 \frac{t^{n-1} dt}{\sqrt{1-t^{2n}}} \text{ and } B_n := \int_0^1 \frac{t^{2n-1} dt}{\sqrt{1-t^{2n}}}.$$

We now show that the quotient L_n/A_n can be evaluated explicitly and that the value of A_n is elementary. This produces an expression for L_n . A similar statement holds for R_n/B_n and B_n .

Observe first that

$$A_n = \int_0^1 \frac{t^{n-1} dt}{\sqrt{1-t^{2n}}} = \frac{1}{n} \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2n} \quad (3.7)$$

and similarly $B_n = 1/n$. Now consider the recursion (3.3) for odd multiples of n to produce

$$A_n = \lim_{m \rightarrow \infty} \prod_{j=1}^{2m} (jn)^{(-1)^j} \times \int_0^1 \frac{t^{(2m+1)n-1} dt}{\sqrt{1-t^{2n}}} \quad (3.8)$$

and similarly the even multiples of n yield

$$B_n = \frac{1}{n} \lim_{m \rightarrow \infty} \prod_{j=1}^{2m+1} (jn)^{(-1)^{j+1}} \times \int_0^1 \frac{t^{2(m+1)n-1} dt}{\sqrt{1-t^{2n}}},$$

in the exact manner as the derivation of (3.5). Therefore using (3.6) and (3.8), and passing to the limit as $m \rightarrow \infty$ so that the integrals disappear, we obtain

$$\frac{L_n}{A_n} = \prod_{j=1}^{\infty} \left[(jn+1)^{(-1)^{j+1}} \times (jn)^{(-1)^{j+1}} \right],$$

so (3.7) yields

$$L_n = \frac{\pi}{2n} \times \prod_{j=1}^{\infty} \left[(jn+1)^{(-1)^{j+1}} \times (jn)^{(-1)^{j+1}} \right].$$

Similarly, using $B_n = 1/n$,

$$R_n = \prod_{j=1}^{\infty} \left[(jn+1)^{(-1)^j} \times (jn)^{(-1)^j} \right].$$

The formula $R_n \times L_n = \pi/2n$ follows directly from here. □

4. CONCLUSIONS

In this paper we have establish that the main radius R_n of the generalized elastic curve (3.2) and the length L_n of this curve satisfy $R_n \times L_n = \frac{\pi}{2n}$. The case $n = 2$ corresponds to the classical Legendre's formula for elliptic integrals.

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