MATH 224 FINAL EXAMINATION – SPRING 2006
Friday, MAY 5 – 3:00- 6:00pm

Your Name:

Your Instructor's Name:

Your Section Number:

INSTRUCTIONS: This exam contains 19 problems. Problem 1 is worth 15 points and has 5 true-or-false questions; for each question, you circle "True" or "False". Problems 2-15 are of multiple-choice, worth 5 points each; circle only one answer (there is only one correct answer for each of these problems). Problems 16-19 are to be graded with partial credits. A table for Laplace transform is attached at the end of the exam.

(1) True or False:

(a) It is possible to find 3 two-dimensional vectors such that they are linearly independent. **True** **False**

(b) Let $\mathbf{y_1}(t)$ and $\mathbf{y_2}(t)$ be two solutions of $d\mathbf{y}/dt = A\mathbf{y}$, where A is a $n \times n$ constant matrix. If $\mathbf{y_1}(0) = 3\mathbf{y_2}(0)$, then $\mathbf{y_1}(t) = 3\mathbf{y_2}(t)$ for all t. **True** **False**

(c) Let A be a 3×3 matrix. If the homogeneous equation $A\mathbf{y} = \mathbf{0}$ has only the trivial solution $\mathbf{y} = \mathbf{0}$, then the inhomogeneous equation $A\mathbf{y} = \begin{pmatrix} 5 \\ 5 \\ 2006 \end{pmatrix}$ has a solution. **True** **False**

(d) If we want to use the following system

$$\begin{cases} \dfrac{dx}{dt} = f(x, y) \\ \dfrac{dy}{dt} = g(x, y) \end{cases}$$

to model the population levels of prey (represented by x) and predators (represented by y), then we should choose f and g such that $\frac{\partial f}{\partial y} > 0$ and $\frac{\partial g}{\partial x} < 0$. **True** **False**

(e) Suppose f and g in the above system are nice (i.e, they and their first order partial derivatives are continuous on the phase plane). If the circle centered at (2, 2) with radius 1 is an orbit, and another solution $(x_1(t),\ y_1(t))$ starts at point (2, 1.5), then we have

$$(x_1(t) - 2)^2 + (y_1(t) - 2)^2 < 1 \text{ for all } t.$$

True **False**

(2) Which of the following is true for the solution of the initial value problem:

$$t\,y' = 2y + 1, \quad y(1) = 1$$

(a) $y(7) = 25$
(b) $y(2) = 0$
(c) $y(3) = 13$
(d) none of these

(3) What is the interval of existence for the solution of the initial value problem:

$$e^t - x\,x' = 0, \quad x(0) = 1$$

(a) $[-2, \infty)$
(b) $(-\ln 2, \infty)$
(c) $(-\infty, -\ln 2]$
(d) $(-\infty, \infty)$

(4) Choose the correct answer for the differential equation

$$y' = y^2 - 4y + 3$$

(a) If $y(0) = 0$, then $y \to \infty$ as $t \to \infty$
(b) If $y(0) = 4$, then $y \to \infty$ as $t \to \infty$
(c) If $y(0) = 3$, then $y \to \infty$ as $t \to \infty$
(d) none of these

(5) A population of fish is growing according to the logistic model with harvesting:

$$\frac{dP}{dt} = P(1 - \frac{P}{100}) - h$$

where $P = P(t)$ is the population level at time t and h is a constant representing the harvesting rate. Suppose that the initial population is 50 fish. If a biologist measures the population of fish after a long time period has elapsed ($t \to \infty$) and finds 60 fish in the lake, what was the harvesting rate h?
(a) 24
(b) 10
(c) 8
(d) none of these

(6) Find the solution of the initial value problem $y'' - 4y' - 5y = \delta(t)$ with $y(0) = y'(0) = 0$.
(a) $y(t) = \frac{1}{3}(e^{5t} + e^{-t})$
(b) $y(t) = \frac{1}{3}(e^{5t} - e^{-t})$
(c) $y(t) = \frac{1}{6}(e^{5t} + e^{-t})$
(d) $y(t) = \frac{1}{6}(e^{5t} - e^{-t})$
(e) None of the above

(7) Find the Laplace transform of the function $f(t)$ which is defined piece-wise by $f(t) = t$ for $0 \leq t < 3$, and $f(t) = 3$ for $t \geq 3$.

(a) $\frac{1}{s}(1 - e^{-2s})$
(b) $\frac{1}{s^2}(1 - e^{-2s})$
(c) $\frac{1}{s}(1 - e^{-3s})$
(d) $\frac{1}{s^2}(1 - e^{-3s})$
(e) None of the above

(8) Find a particular solution for the differential equation $y'' - 3y' - 10y = 21e^{-2t} - 50$.

(a) $y_p(t) = 5 - te^{-2t}$
(b) $y_p(t) = -5 + 3te^{-2t}$
(c) $y_p(t) = -5 - 3te^{-2t}$
(d) $y_p(t) = 5 - 3te^{-2t}$
(e) None of the above

(9) Consider the matrix

$$A = \begin{pmatrix} -3 & 0 & 0 \\ -5 & 6 & -4 \\ -5 & 2 & 0 \end{pmatrix}$$

Then the *smallest* eigenvalue λ_1 and a corresponding eigenvector $\mathbf{v}_1$, respectively, are

(a) $\lambda_1 = 0$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) $\lambda_1 = -3$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(c) $\lambda_1 = -3$, $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(d) None of the above

(10) Let

$$A = \begin{pmatrix} 2 & -1 & 3 & 4 \\ 0 & 2 & -2 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 3 & 1 & 0 \end{pmatrix}$$

(a) $det\,(A) = 0$ and therefore, A is nonsingular.
(b) $det\,(A) = 16$ and therefore, A is nonsingular.
(c) $det\,(A) = -16$ and therefore, A is nonsingular
(d) $det\,(A) = 4$ and therefore, A is singular.

(11) Let $\mathbf{y}(t)$ be the solution of the system

$$\mathbf{y}' = \begin{pmatrix} 1 & 0.25 \\ -1 & 1 \end{pmatrix} \mathbf{y}$$

with the initial condition $\mathbf{y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Then $\mathbf{y}(2\pi)$ is (to save your time, use the information: $1 + 0.5i$ is an eigenvalue, and $\begin{pmatrix} 1 \\ 2i \end{pmatrix}$ is a corresponding eigenvector)

(a) in the first quadrant ($y_1(2\pi) > 0,\ y_2(2\pi) > 0$)
(b) in the second quadrant ($y_1(2\pi) < 0,\ y_2(2\pi) > 0$)
(c) in the third quadrant ($y_1(2\pi) < 0,\ y_2(2\pi) < 0$)
(d) in the fourth quadrant ($y_1(2\pi) > 0,\ y_2(2\pi) < 0$)
(e) $\mathbf{y}(2\pi) = \mathbf{0}$.

(12) Let $\mathbf{y}(t)$ be the solution of the system

$$\mathbf{y}' = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{y}$$

with the initial condition $\mathbf{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $\mathbf{y}(2\pi)$ is

(a) in the first quadrant ($y_1(2\pi) > 0,\ y_2(2\pi) > 0$)
(b) in the second quadrant ($y_1(2\pi) < 0,\ y_2(2\pi) > 0$)
(c) in the third quadrant ($y_1(2\pi) < 0,\ y_2(2\pi) < 0$)
(d) in the fourth quadrant ($y_1(2\pi) > 0,\ y_2(2\pi) < 0$)
(e) $\mathbf{y}(2\pi) = \mathbf{0}$.

(13) For which of the following systems $\mathbf{y}' = A\mathbf{y}$ is the point $\mathbf{0}$ a spiral source?

(a) $A = \begin{pmatrix} 2 & 4 \\ 7 & -1 \end{pmatrix}$
(b) $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$
(c) $A = \begin{pmatrix} 3 & 2 \\ 8 & -3 \end{pmatrix}$
(d) $A = \begin{pmatrix} -1 & 0 \\ 5 & -2 \end{pmatrix}$.

(14) Let A be a 3×3 real matrix. In which one of the following cases is $\mathbf{0}$ an asymptotically stable equilibrium point of the system $\mathbf{y}' = A\mathbf{y}$?

(a) The eigenvalues of A are $-3,\ -2+2i,\ -2-2i$
(b) The eigenvalues of A are $-3,\ 2i,\ -2i$
(c) The eigenvalues of A are $-3,\ -2,\ 1$
(d) The eigenvalues of A are $1,\ 2,\ 3$.

(15) The general solution of the system

$$\begin{cases} \dfrac{dx}{dt} = -x - \ y \\ \dfrac{dy}{dt} = \ \ x - 3y \end{cases}$$

is:

(a) $C_1 e^{-2t}\begin{pmatrix}1\\1\end{pmatrix} + C_2 e^{-2t}\begin{pmatrix}1\\0\end{pmatrix}$

(b) $C_1 e^{-2t}\begin{pmatrix}-1\\-1\end{pmatrix} + C_2 t e^{-2t}\begin{pmatrix}-1\\0\end{pmatrix}$

(c) $C_1 t e^{-2t}\begin{pmatrix}1\\1\end{pmatrix} + C_2 e^{-2t}\begin{pmatrix}0\\1\end{pmatrix}$

(d) $C_1 e^{-2t}\begin{pmatrix}-1\\-1\end{pmatrix} + C_2\left(t e^{-2t}\begin{pmatrix}1\\1\end{pmatrix} + e^{-2t}\begin{pmatrix}1\\0\end{pmatrix}\right)$

(e) none of the above.

(16) (10 points) A 1-kg mass is attached to a spring with $k = 4kg/s^2$ and the system is allowed to come to rest. The spring-mass system is attached to a machine that supplies an external driving force $f(t) = 6\sin t$ Newtons. The system is started from equilibrium with the mass having no initial displacement nor velocity. Disregard any damping forces. Find the position of the mass as a function of time.

(17) (10 points) Use the Laplace transform to solve the second order initial value problem $y'' - 9y = -2e^t$, $y(0) = 0$, $y'(0) = 1$.

(18) (15 points) A tank initially holds 100 gallons of a brine solution containing 1 pound of salt. At $t = 0$ another brine solution containing 1 pound of salt per gallon is poured into the tank at the rate of 3 gallons per minute, while the well-stirred mixture leaves the tank at the same rate. Find the approximate time at which the mixture in the tank contains 25 pounds of salt.

(19) (20 points) Consider the nonlinear system

$$\begin{cases} \dfrac{dx}{dt} = y - x \\ \dfrac{dy}{dt} = -(3 - x - 2y)\,x. \end{cases}$$

(i) Find and draw x- and y- nullclines on the phase plane; indicate which is which.

(ii) Find all equilibrium points of the system.

(iii) The phase plane is divided by the nullclines into several regions. In each region, draw an arrow to indicate the general direction of the vector field.

(iv) Draw in the phase plane at least one non-equilibrium solution.

(v) Determine the type (saddle, nodal source, center, etc.) of the equilibrium point in the interior of the first quadrant.

Frequently encountered Laplace transforms:

The original	The Laplace transform
$f(t) \equiv C = \text{constant}$	$\mathcal{L}[f(t)] = \frac{C}{s}, \quad s > 0$
$f(t) = e^{at}$	$\mathcal{L}[f(t)] = \frac{1}{s-a}, \quad s > a$
$f(t) = t^n$	$\mathcal{L}[f(t)] = \frac{n!}{s^{n+1}}, \quad s > 0$
$f(t) = t^n e^{at}$	$\mathcal{L}[f(t)] = \frac{n!}{(s-a)^{n+1}}, \quad s > a$
$f(t) = \sin kt$	$\mathcal{L}[f(t)] = \frac{k}{s^2+k^2}, \quad s > 0$
$f(t) = \cos kt$	$\mathcal{L}[f(t)] = \frac{s}{s^2+k^2}, \quad s > 0$
$f(t) = e^{at}\sin kt$	$\mathcal{L}[f(t)] = \frac{k}{(s-a)^2+k^2}, \quad s > a$
$f(t) = e^{at}\cos kt$	$\mathcal{L}[f(t)] = \frac{s-a}{(s-a)^2+k^2}, \quad s > a$
$f(t) = H_a(t) = H(t-a)$	$\mathcal{L}[f(t)] = \frac{e^{-as}}{s}, \quad s > 0$
$f(t) = \delta_a(t)$	$\mathcal{L}[f(t)] = e^{-as}, \quad s > 0$

Rules for the Laplace transform:

The Laplace transform and the inverse Laplace transform are linear: $\mathcal{L}[\alpha f(t)](s) = \alpha \cdot \mathcal{L}[f(t)](s)$ if α is a constant, and $\mathcal{L}[f(t)+g(t)](s) = \mathcal{L}[f(t)](s) + \mathcal{L}[g(t)](s)$; the same for the inverse: $\mathcal{L}^{-1}[\alpha F(s)](t) = \alpha \cdot \mathcal{L}^{-1}[F(s)](t)$, $\mathcal{L}^{-1}[F(s)+G(s)](t) = \mathcal{L}^{-1}[F(s)](t) + \mathcal{L}^{-1}[G(s)](t)$.

For the Laplace transform: $\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f(t)](s-a)$, for the inverse: $\mathcal{L}^{-1}[F(s-a)](t) = e^{at}\mathcal{L}^{-1}[F(s)](t)$;

For the Laplace transform: $\mathcal{L}[H_a(t)f(t-a)](s) = e^{-as}\mathcal{L}[f(t)](s)$, for the inverse: $\mathcal{L}^{-1}[e^{-as}F(s)](t) = H_a(t) \cdot \mathcal{L}^{-1}[F(s)](t-a)$.

The Laplace transform for the derivatives: $\mathcal{L}\left[\frac{df}{dt}\right](s) = s\mathcal{L}[f(t)](s) - f(0)$.

$\mathcal{L}\left[\frac{d^2f}{dt^2}\right](s) = s^2\mathcal{L}[f(t)](s) - sf(0) - f'(0)$.

The derivative of Laplace transform: $\frac{d}{ds}\mathcal{L}[f(t)](s) = -\mathcal{L}[tf(t)](s)$.