

Mathematics 305-01  
Final Exam  
May 5, 2006  
Answer all Questions

2 points each Define the following:

- a) Open set of real numbers
- b) Compact set of real numbers
- c) Convergent sequence of real numbers
- d) Continuous function
- e) Uniformly continuous function
- f) Differentiable function
- g) Integrable function
- h) Absolutely convergent improper integral
- i) Uniformly convergent sequence of functions
- j) Convergent infinite series of real numbers

10 points Show that

$$\lim_{x \rightarrow 2} x^3 = 8.$$

10 points What is

$$\lim_{n \rightarrow \infty} n (\sqrt{n^2 + 2} - \sqrt{n^2 + 1})?$$

Prove your answer.

5 points each a) State the Mean Value Theorem.

b) Suppose that  $f$  is a differentiable function on the interval  $(a, b)$  and that for each  $x \in (a, b)$   $f'(x) = 0$ . Show that  $f(x) = \text{constant}$ .

10 points Suppose that  $m \leq f(x) \leq M$  for all  $x \in [a, b]$  and that  $f(x)$  is integrable on  $[a, b]$ . Show that

$$m(b - a) \leq \int_a^b f(x) dx \leq M(b - a).$$

10 points Show that the function

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$$

is not integrable on the interval  $[0, 1]$ . You should use the definition of integrability from problem 1g).

10 points Show that an absolutely convergent improper integral is convergent. You should use the definition of absolute convergence from 1h).

10 points each a) Given an example of a sequence of functions  $f_n(x)$  defined on an interval  $I$  which converge at each  $x \in I$  to a function  $f(x)$ , but which do not converge uniformly to  $f$  on  $I$ . Prove your answer.

b) Prove that if  $f_n(x)$  is a sequence of integrable functions which converge uniformly to an integrable function  $f(x)$  on the interval  $[a, b]$ , the

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx.$$

3 points each part For each of the following infinite series, tell whether it converges absolutely or conditionally or diverges. Give reasons for your assertions:

i

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$$

ii

$$\sum_{k=1}^{\infty} e^{-k}$$

iii

$$\sum_{k=1}^{\infty} \frac{k}{k+2}$$

iv

$$\sum_{k=1}^{\infty} \frac{2^k k!}{k^k}$$

v

$$\sum_{k=1}^{\infty} \frac{\sin k}{k}$$

vi

$$\sum_{k=1}^{\infty} \frac{1}{k^{1+1/k}}$$