1. (15 points) Compute the following limits:

(a)

$$\lim_{h\to 0} \frac{\frac{1}{\sqrt{4+h}} - \frac{1}{2}}{h}$$

(b)

$$\lim_{\theta\to 0} \frac{\sin(2\theta)}{\tan(\theta)}$$

(c)

$$\lim_{x\to\infty} x^{1/x}$$

2. (20 points) Find $\frac{dy}{dx}$ when:

(a)

$$y = \sqrt{x + \sqrt{x}}$$

(b)

$$y = \frac{xe^x}{1 + \ln x}$$

(c)

$$y = \sec^2\left(\sqrt[3]{x}\right)$$

(d)

$$(x^2 + y^2)^2 = x^2 - y^2$$

3. (15 points) Let

$$f(x) = \begin{cases} x^2 + 1, & \text{if } x \leq 1; \\ cx, & \text{if } x > 1. \end{cases}$$

(a) For what value of c is $f(x)$ continuous at $x = 1$?
(b) For the value of c that you found in part (a), is $f(x)$ differentiable at $x = 1$? Justify your answer.

4. (10 points) A paper cup has the shape of a cone with height 8 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm/s, how fast is the water level rising when the water is 4 cm deep?

5. (15 points) Find the points on the ellipse $x^2 + 4y^2 = 4$ that are closest to the point $(1, 0)$. (Hint: Minimize the square of the distance between the points.)

6. (20 points) Consider the function

$$f(x) = \frac{1}{x(x-3)^2}.$$

Its derivatives are

$$f'(x) = \frac{-3(x-1)}{x^2(x-3)^3}$$

and

$$f''(x) = \frac{6(2x^2-4x+3)}{x^3(x-3)^4}.$$

(a) What is the domain of $f(x)$? Find all horizontal and vertical asymptotes of the graph of $y = f(x)$.
(b) On what intervals is $f(x)$ increasing, and on what intervals is $f(x)$ decreasing? Find all local maxima and local minima of $f(x)$.
(c) On what intervals is $f(x)$ concave up, and on what intervals is $f(x)$ concave down? Find all inflection points of $f(x)$.
(d) Use the information from parts (a) through (c) to draw a sketch of the curve $y = f(x)$.

7. (15 points) (a)If

$$F(x) = \int_0^x e^{-t^2}\, dt$$

find $F'(x)$.
(b) If

$$G(x) = \int_0^{\sqrt{x}} e^{-t^2}\, dt$$

Find $G'(x)$.
(c) Evaluate the definite integral

$$\int_1^4 x^{-3/2}\, dx$$

8. (15 points) Find the volume of the solid generated by rotating the circle of radius 1 centered at the point $(2, 0)$ around the y-axis.

9. (20 points) Evaluate the indefinite integrals:

(a)

$$\int x^2 \sin(x^3 + 1)\, dx$$

(b)

$$\int e^{3x} \sin(x)\, dx$$

(c)

$$\int \frac{x}{1 - x^2}\, dx$$

(d)

$$\int \sin^3 x\, dx$$

(e)

$$\int \frac{\sin x dx}{\sqrt{\cos x + 1}}$$

10. (15 points) Find the area of the surface generated by rotating the curve

$$y = \frac{x^2}{4} - \frac{\ln(x)}{2}$$

for $1 \leq x \leq e$ around the x axis

11. (15 points) This problem will concern the differential equation

$$y' = y(3 - y).$$

a) Draw direction fields for the equation, and use them to sketch the graphs of solutions to the equation, with initial conditions $y(0) = 1$ and $y(0) = 4$.
b) Use Euler's method with step size .25 to find an approximation to $y(1)$ for the initial condition $y(0) = 1$.
c) Find a formula for the solution to the equation.

12 (20 points) a)Sketch a graph of the polar curve

$$r = 3\cos\theta.$$

b) Sketch a graph of the polar curve

$$r = 1 + \cos\theta.$$

c) Find the slope of the curve in part b) at the point $(r, \theta) = (3/2, \pi/3)$.
d) Find the area between the curves.

13. (15 points) Find the limit of each of the following sequences and justify your answer.

a)

$$a_n = \frac{n^2 + 4}{n^3 - 1}$$

b)

$$b_n = \sqrt{n^2 + 1} - n$$

c)

$$c_1 = 1 \quad c_{n+1} = \frac{1}{2}(c_n + 4)$$

14. (18 points) For each of the following infinite series, tell whether it converges or diverges. If it converges, tell whether it converges absolutely or conditionally. Justify your answers.

a)

$$\sum_{n=1}^{\infty} \frac{n^2+4}{2n^2-9}$$

b)

$$\sum_{n=1}^{\infty} 4^{-n}$$

c)

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

d)

$$\sum_{n=1}^{\infty} \frac{n}{n^2+3n+8}$$

e)

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

f)

$$\sum_{n=1}^{\infty} \frac{e^n}{n!}$$

15. (10 points) Estimate the error is using

$$\frac{1}{1^3} - \frac{1}{2^3} + \cdots - \frac{1}{8^3}$$

as an approximation to the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}.$$

16. (12 points) Find the radius of convergence of each of the following power series:

a)

$$\sum_{n=1}^{\infty} \frac{x^n}{n^4}$$

b)

$$\sum_{n=1}^{\infty} n^n x^n$$

c)

$$\sum_{n=1}^{\infty} \sqrt{n+1} x^n$$

17. (25 points) (a)Write down the Taylor series about $x = 0$ for the function

$$f(x) = \frac{\sin x - x}{x^3}.$$

and use the series to write

$$\int \frac{\sin x - x}{x^3}$$

as an infinite series.

(b) Write out the Taylor series for the function

$$e^{-x^2}$$

about $x = 0$.

(c) Estimate the error in evaluating e^{-4} by using the series in part (b) and stpping at the x^2 term.