Math 221 Final
December 11, 2007

Name: ______________________
Instructor: ______________________
Section: ______________________

INSTRUCTIONS — READ THIS NOW

- This test has 14 problems on 15 pages (including this one) worth a total of 250 points.
- Write your name, your instructor's name, and your section number **right now**.
- Show your work. To receive full credit, the work must be clearly shown. If you need more space, write on the back side of the paper, but be sure to clearly label your work.
- Please turn off all cellphones, audio recording/playing devices and any other electronic device that could disrupt others taking the test around you.
- This is a 4 hour test. Read the questions carefully and simplify your answers when you are asked to do so.

FORMULAS:

- $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$
- $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos(2x) = \cos^2(x) - \sin^2(x)$

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1. (**5 points each**) TRUE or FALSE. Please write TRUE or FALSE to the left of each statement. No work or explanations are needed in this question. Vectors and vector functions mentioned are assumed to be 3-dimensional unless otherwise stated.

   (a) The vector $(\mathbf{v} \times \mathbf{u})$ is always parallel to $\mathbf{u}$.

   (b) Suppose you are given three non-zero vectors $\mathbf{v}$, $\mathbf{u}$, and $\mathbf{w}$ with $\mathbf{v} \times \mathbf{u} = \mathbf{v} \times \mathbf{w}$. Then $\mathbf{u} = \mathbf{w}$.

   (c) The following integral represents the area of the surface S.
   $$\iint_S 1\, dS$$

   (d) If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S then the surface integral
   $$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$
   is the flux of $\mathbf{F}$ across S.

   (e) Suppose C is a smooth curve lying in the plane and the curvature of C is 0 everywhere. Then C is a line.

   (f) Suppose you are driving your car around an parabolic course and your speedometer reads a constant 42 mph. Then your acceleration is zero.

   (g) You are given a differentiable scalar function $f(x, y, z)$. The direction of steepest ascent at the point (a, b, c) is perpendicular to $\nabla f(a, b, c)$.

   (h) Suppose you have a curve C defined by the vector function $\mathbf{r}(t)$. Then the unit tangent vector $\mathbf{T}(t)$ and the unit binormal vector $\mathbf{B}(t)$ are perpendicular to each other.

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2. (**5 points each**) Evaluate the following integrals:

   (a) $\displaystyle\int_0^{2\pi} \langle \cos^2 t, e^t, t^4 \rangle \, dt$

   (b) $\displaystyle\int_0^2 \int_0^1 3xe^{2y} \, dy \, dx$

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3. (**5 points each**) Evaluate the following:

   (a) The unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$ corresponding to the curve $\mathbf{r}(t) = \langle t, 1 + t^2, t \rangle$.

   (b) The directional derivative of $f(x, y) = x^2y - 3xy^3$ at the point $(1, -1)$ in the direction of the vector $\mathbf{v} = \langle 1, 1 \rangle$.

   (c) $\partial z/\partial t$ where $z = 2x + 3y$, $x = \sin 2t$ and $y = \cos t$.

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4. (**5 points each**) Given the two functions $f(x, y) = 3x^2 + 2y^2 + 8x - 15$ and $g(x, y) = 2x^2 - 3y^2 - 12y + 38$

   (a) Find equation of the tangent plane of $z = f(x, y)$ at the point $(1, 2, 4)$. Simplify your answer.

   (b) Find equation of the tangent plane of $z = g(x, y)$ at the point $(1, 2, 4)$. Simplify your answer.

   (c) Find the angle between the two tangent planes. (You do not need to simplify.)

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5. (**20 points**) Assume b is a positive number. Find the arclength of the curve C defined by the vector function

$$\mathbf{r}(t) = \langle -3\sin t, b\,t, 3\cos t\rangle$$

over the interval $-10 \leq t \leq 10$.

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6. (**5 points each**) The equation $x^3 + y^3 + z^3 + 6xyz = 22$ defines y implicitly as a function of x and z in a neighborhood around the point $(1, 1, 2)$.

   (a) Find $\dfrac{\partial y}{\partial x}$ and $\dfrac{\partial y}{\partial z}$ as functions of x, y, and z.

   (b) Evaluate these partial derivatives at the point $(1, 1, 2)$.

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7. (**10 points each**) Evaluate the integrals:

   (a) $\iint_D \sqrt{1+y^3}\, dA$ where D is the region in the first quadrant bounded by $x = y^2$, $x = 0$ and $y = 2$.

   (b) $\int_C (2x^2y + 5y^2)\, dx - 2xy^2\, dy$ where C is the circle of radius 2 centered at the origin of the xy-plane.

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8. (**15 points**) Find the minimum of the function

$$f(x, y) = x^2 + 2y^2 + 2xy + 2x + 3y$$

subject to the condition that $x^2 - y = 1$.

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9. (**20 points**) Let $\mathbf{F} = \langle -2x + y + z, x + 3y - 4z, -x - y + 5z \rangle$ and suppose the closed surface S and the region E it encloses satisfy the hypotheses of the Divergence Theorem. Show that the volume of the region E is

$$V = C \iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

and give the value of the constant C.

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10. (**15 points**) Find the volume of the following solid, described in spherical coordinates:

$$E = \{(\rho, \varphi, \theta) : 1 \leq \rho \leq 3, \pi/6 \leq \varphi \leq \pi/4, 0 \leq \theta \leq \pi/4\}.$$

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11. (**20 points**) Find the area of the surface cut from the hemisphere $S := \{(x, y, z) : x^2 + y^2 + z^2 = 4, z \geq 0\}$ by the cylinder $C := \{(x, y, z) : x^2 + y^2 = 2x, z \geq 0\}$. (Hint: Use cylindrical coordinates.)

12. (**5 points each**) Given the vector field $\mathbf{F} = \langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3 + \cos z \rangle$.

    (a) Show that $\mathbf{F}$ is conservative.

    (b) Find a potential function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.

    (c) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where the curve C is the line from $(1, 1, 0)$ to $(2, 1, \pi)$.

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13. (**20 points**) Evaluate the surface integral

$$\iint_S \sqrt{x}\, dS$$

where S is the surface given parametrically by $x = u^2$, $y = v$, and $z = 4u + v$ with $0 \leq u \leq 1$ and $0 \leq v \leq 3$.

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14. (**5 points/10 points**)

    (a) Given the vector function $\mathbf{F}(x, y, z) = \langle\, z^2, y^2, x\rangle$, find the curl of $\mathbf{F}$.

    (b) Use Stokes' theorem to evaluate

    $$\int_C \mathbf{F} \cdot d\mathbf{r}$$

    where C is the triangle (in 3D) with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

STOP. THIS IS THE LAST PAGE.