Math 309/609

Linear Algebra Final Exam

Fall 2007

Name: _____

Soc. Sec. No.: _____

Signature: _____

Note: no credit will be given if your work is not shown!

For the following three problems,

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 1 & 0 & -1 & 2 \\ 1 & -2 & 3 & -4 \end{bmatrix}$$

1. Find a basis for the nullspace of A.

2. Find an orthogonal basis for the orthogonal complement of the row space of A.

3. Let $\vec{b} = (1, -3, a)^t$. Give conditions on a so that $A\vec{x} = \vec{b}$ has

(a). a unique solution.

(b). more than one solution.

(c). no solution.

4. Let

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 2 & -4 & 0 \\ 2 & -1 & -4 \end{bmatrix}.$$

(a). Find the trace of A.

(b). Find the characteristic polynomial of A.

(c). Find all of the eigenvalues for A.

(d). For each eigenvalue of A, find a basis for its eigenspace.

(e). For each eigenvalue of A, find both of its algebraic and geometric multiplicities.

(f). Is A diagonalizable? If A is diagonalizable, then find a matrix S and a diagonal matrix D such that $A = SDS^{-1}$. If A is not diagonalizable, explain why?

5. Let V be the 4-dimensional vector space of polynomials of degree three (3) or less. Consider the linear transformation $L : V \to V$ defined by $L(p) = xp'(x) - 2p(x)$.

(a). Find the matrix A which represents L in the standard basis $\{1, x, x^2, x^3\}$ for V.

(b). Find a basis for the nullspace of L.

(c). Find a basis for the range space of L.

(d). What is the rank of L?

(e). What is the nullity of L?

(f). What is the rank of A?

(g). What is the dimension of the nullspace of A?

(h). Find the characteristic polynomial for L.

(i). Find all of the eigenvalues for L.

(j). Is L diagonalizable? If L is diagonalizable, then find a basis $(\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4})$ for V and a diagonal matrix D such that $L(\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}) = (\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4})D$. If L is not diagonalizable, explain why?

6. Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix}$$

(a). Find $\det(A)$.

(b). If A is nonsingular, find its inverse. If A is singular, explain why?

(c). Find its $(3,1)$-cofactor A_{31} for A.

(d). Find the product matrix $A(adjA)$.

7. Let A be an $m \times n$ matrix of real numbers. If the dimension of the nullspace of A is k, compute the following in terms of k and the dimension of A.

(a). $\dim \mathcal{R}(A)$.

(b). $\dim \mathcal{R}(A^t)$.

(c). $\dim \mathcal{N}(A)$.

(d). $\dim \mathcal{N}(A^t)$.

(e). Find an orthogonal decomposition for both R^m and R^n in terms of the four fundamental subspaces associated with A.

8. Let $\vec{u}$ be a given column vector in R^n and $|\vec{u}|^2 = 2$. Consider the matrix $R = I - \vec{u}\vec{u}^t$.

(a). Show that R is an orthogonal matrix.

(b). Compute $R\vec{u}$.

(c). If $\vec{v}$ is orthogonal to $\vec{u}$, compute $R\vec{v}$.

(d). Find all of the eigenvalues for the matrix R and their corresponding algebraic and geometric multiplicities.

9. Let $L : R^3 \rightarrow R^3$ be the linear transformation defined by the reflection through the plane $P = \{(x, y, z) \in R^3 | x - y = 0\}$.

(a). Find an orthonormal basis for P.

(b). Find an orthonormal basis for $P^{\perp}$.

(c). Find an orthonormal basis such that the matrix representation of L in this orthonormal basis is a diagonal matrix.

(d). Find the matrix A that represents L with respect to the standard basis for R^3.

(10). Let A and B be two $n \times n$ matrices.

(a). Define A to be similar to B.

(b). Define A to be congruent to B.

(c). Is a real symmetric matrix always similar to a diagonal matrix? Why?

(d). Is a real symmetric matrix always congruent to a diagonal matrix? Why?

11. Let $\vec{z} = (x, y)^t$ and $Q(\vec{z}) = 23x^2 - 72xy + 2y^2$.

(a). Find the symmetric bilinear form B associated with the quadratic form Q.

(b). Find a symmetric matrix A such that $Q(\vec{z}) = \vec{z}^* A\vec{z}$.

(c). Find an orthogonal matrix S such that A is congruent to a diagonal matrix D via S.

(d). Use the principal axes theorem to diagonalize the quadratic form Q.

(e). Does Q has a local extrema at the origin? Explain.

12. State the spectral theorem for a self-adjoint linear transformation from a finite dimensional real product space to itself.

13. Let

$$A = \begin{bmatrix} -2 & 1 & 0 & -1 \\ 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -3 \end{bmatrix}$$

(a). Find a basis for each of the four fundamental subspaces associated with A.

(b). Find the rank of A, the nullity of A, and the nullity of A^t.

(c). Find the dimension of the row space of A and the dimension of the column space of A.

(d). Are the three row vectors of A linearly independent? Justify your answer by either prove they are linearly independent or provide an explicit nontrivial linear relation of the three row vectors.

(e). Is $\mathcal{N}(\mathcal{A})$ isomorphic to $\mathcal{R}(\mathcal{A})$? Is $\mathcal{R}(\mathcal{A})$ isomorphic to $\mathcal{R}(\mathcal{A}^t)$? Is $\mathcal{N}(\mathcal{A})$ isomorphic to $\mathcal{N}(\mathcal{A}^t)$?