Linear Algebra and Advanced Calculus Examination

April 8, 1988

1. Let V be an n-dimensional vector space over $\mathbb{R}$. Define the dual space V^* of V, and prove that V^* is also an n-dimensional vector space over $\mathbb{R}$.

2 a) Let V be an n-dimensional vector space over $\mathbb{R}$, and let $L : V \to V$ be a linear transformation. Define the notions of eigenvalue and eigenvector for L

b) Suppose in particular that $L^2 = L$ (i.e. $L(Lv) = Lv, \forall v \in V$). Prove that 0 and 1 are the only eigenvalues of L, and that every $v \in V$ can be written uniquely in the form $v = u + w$, where u is in the null space (or kernel) of L and w is in the range of L. [Hint: try $w = Lv$.] What are the eigenvectors associated with the eigenvalue 1?

3 a) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a function, and let $p \in \mathbb{R}^n$. What does it mean to say that F is differentiable at p? Prove that if F is differentiable at p, then F is continuous at p.

b) If the real-valued component functions of F have continuous first partial derivatives, describe the differential dF as a matrix,

1. Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $F(x,y) = (e^x \cos y, e^x \sin y)$. By stating and using a general theorem, show that F has a local inverse near each $p \in \mathbb{R}^2$. Show also that F has no single inverse defined for all $p \in \mathbb{R}^2$.

5 Let C be the unit circle $\{(x,y,0) \mid x^2+y^2=1\}$ in the x,y plane, oriented counterclockwise. Let D be the disc $\{(x,y) \mid x^2+y^2 \leq 1\}$. Let S be the hemispherical surface with equation $z = \sqrt{1-x^2-y^2}$, parametrized by $\sigma(x,y) = (x, y, \sqrt{1-x^2-y^2})$ for $(x,y) \in D$. Formulate Stokes' Theorem and use it to evaluate

$$\oint_C (y e^x dx + e^x dy + dz).$$