LINEAR ALGEBRA AND VECTOR CALCULUS

10 JANUARY 1990

Note: You should state carefully any results which you use.

1. Suppose that T is a linear transformation from a real n-dimensional vector space, V, to itself. Let $\{v_1, v_2,...,v_n\}$ and $\{w_1,w_2,...,w_n\}$ be two bases of V with

$$w_k=\sum_{i=1}^{n} a_{ik}v_i$$

for a nonsingular matrix $A=(a_{ik})$.Suppose that the matrix representing T with respect to the basis $\{v_1, v_2,...,v_n\}$ is (t_{rs}), find the matrix representing T with respect to the basis $\{w_1,w_2,...,w_n\}$.

2. a. Find the rank of the matrix

$$A=\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

Your answer should include a definition of the rank of a matrix.

b. Find a basis for the null-space of A.

3. A real matrix,Q, is called orthogonal if $Q^t = Q^{-1}$.

a. Show that the eigenvalues of an orthogonal matrix.have absolute value 1.

b. Show that the determinant of an orthogonal matrix is ±1.

c.Show that a 2x2 matrix of determinant 1 is orthogonal if and only if it is of the form

$$Q_t=\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

4. Consider the transformation of R^2 to itself given by

$$(x,y) \xrightarrow{F} (x2 + y2,\ -2xy).$$

a. Where does F have a local differentiable inverse?

b. Where is F locally one to one?

c. Is F one to one on R^2?

5. Let f be continuous in $R = \{x^2+y^2+z^2 \leq 1\}$. Show that

$$\int\int\int_R f(x,y,z)dxdydz = \int\int\int_D f(r\cos\theta \sin\phi, r\sin\theta \sin\phi, r\cos\phi)r^2\sin\phi \, dr d\theta \, d\phi$$

where D is a domain in R^3. Identify D.

6. Show that the area, A, of a bounded region, R, in R^2 with smooth boundary, S, is given by

$$A = 1/2\int_S -ydx+xdy.$$