Linear Algebra Exam

Tuesday, March 6, 1990

1:30 P.M.

1. Let $v_1 = \binom{1}{0}$, $v_2 = \binom{0}{1}$ and $w_1 = \binom{1}{1}$, $w_2 = \binom{1}{-1}$ be two bases for $\mathbf{R}^2$.

   Let $T\binom{x}{y} = \binom{3x+y}{x-2y}$ for $\binom{x}{y} \in \mathbf{R}^2$.

   a) Find the matrix of T with respect to $\{v_1, v_2\}$.

   b) Find the matrix of T with respect to $\{w_1, w_2\}$.

2. Suppose that T is a linear transformation from a real n-dimensional vector space, V, to itself. Let $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ be two bases of V with

$$w_k = \sum_{i=1}^{n} a_{ik} v_i$$

   for a nonsingular matrix $A = (a_{ik})$. Suppose that the matrix representing T with respect to the basis $\{v_1, v_2, \ldots, v_n\}$ is (t_{rs}), find the matrix representing T with respect to the basis $\{w_1, w_2, \ldots, w_n\}$.

3. Find all eigenvalues and eigenvectors of

$$T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

4. Let $Q = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for $\theta \in \mathbf{R}$.

   Show that

   a) Q is invertible

   b) $Q^{-1} = Q^t$ (transpose)

   c) det Q=1

   d) Calculate the eigenvalues and eigenvector of Q.

   e) Let P be an $n \times n$ matrix satisfying $P^t = P^{-1}$. What can you conclude about P?