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# Linear Algebra and Advanced Calculus Examination

1. Define the following terms.

(a) $v_1, \ldots, , v_n$ are a basis for the vector space $V$.

(b) orthogonal matrix

(c) the dual $V^*$ of a real vector space $V$.

(d) an eigenvector for a linear transformation $T : V \to V$.

2. Suppose $L$ is the linear transformation from $\mathbf{R}^3 \to \mathbf{R}^3$ which reflects each vector through the plane $x + y + z = 0$.

(a) Find an orthonormal basis $\mathcal{B} = q_1, q_2, q_3$ of $\mathbf{R}^3$ so that $q_1, q_2$ is a basis for this plane.

(b) Find a matrix $C$ which represents $L$ with respect to this basis $\mathcal{B}$.

(c) Find the matrix $A$ which represents $L$ with respect to the standard basis $e_1, e_2, e_3$ of $\mathbf{R}^3$.

(d) Where does the reflection send the vector $(1, 2, 1)$?

3. Let $A = \begin{pmatrix} .55 & .45 \\ .45 & .55 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of $A$.

(b) Give an orthogonal matrix $P$ and a diagonal matrix $D$ with $A = PDP^{-1}$.

(c) Compute $A^{20}$ to within $10^{-10}$.

(d) Sketch the graph of $55x^2 + 90xy + 55y^2 = 100$.

(Hint: Use the previous parts to change coordinates via a rotation so that the equation has no mixed term in the new coordinates.)

(e) If $f : \mathbf{R}^2 \to \mathbf{R}$ is a differentiable function with zero first partial derivatives at $(1, 0)$ and whose Hessian matrix of second order partial derivatives at $(1, 0)$ is given by $A$, what can one say about whether the function has a local minimum, local maximum, or neither at $(1, 0)$.

4. Determine whether each of the following is true or false. Give a *brief* justification for each answer.

(a) If $A$ is a 3 by 3 matrix with eigenvalues 0,1,1, then $A$ is diagonalizable.

(b) If $A$ is a 3 by 4 matrix of rank 2, then $A^tA$ has 0 as an eigenvalue of geometric multiplicity 2, i.e. the dimension of the eigenspace is 2.

(c) If $\det A = 0$, then $A$ is invertible.

(d) The eigenvalues of an orthogonal matrix must be real numbers.

(e) If 5 vectors span a 5 dimensional vector space $S$, then they form a basis for $S$.

(f) A projection matrix ( i.e. one which projects orthogonally onto a subspace) has only 0 and 1 as eigenvalues.

5. Find the absolute maximum and absolute minimum of $f(x, y) = xy^2 + x^2$ on the set $U = \{(x, y) : x^2 + y^2 \leq 1\}$.

6. Suppose $\gamma(t) = (t, t^2)$ is a contour line for a function $f(x, y)$; i.e. $f(\gamma(t)) = c$. Show that the gradient $\nabla f(1, 1)$ is a multiple of the vector $(2, -1)$.

7. (a) State the change of variables formula for a function $f : A \subset R^n \to B = f(A) \subset R^n$ which expresses $\int_B g(y_1, \cdots, y_n)\, dV$ in terms of an integral over $A$ under certain hypotheses.

(b) Use the change of variables formula to evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \exp(x^2 + y^2)\, dx\, dy.$$

8. (a) Let $\gamma$ be a parametrization of $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$. Find $\int_\gamma y\, dx + x\, dy$.

(b) State Green's theorem.

(c) Use Green's theorem to find the line integral of the vector field $F(x, y) = (2x - y, x + 2y)$ over the boundary of the region bounded by $y = 1 - x^2$ and the $x-$ axis traced in the counterclockwise direction.

9. Consider the surface $\sigma : S \to \mathbf{R}^3$ where $S = \{(u, v) : u^2 + v^2 \leq 1\}, \sigma(u, v) = (u, v, \sqrt{4 - u^2 - v^2})$.

(a) Find the area of the surface.

(b) Find the rate of flow of the constant vector field $(0, 0, 1)$ across the surface.

(c) Let $\gamma(t) = (\cos t, \sin t, \sqrt{3})$. Find $\int_\gamma x\, dx + y\, dy + z\, dz$ both by directly evaluating the integral and by applying Stokes theorem.