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Linear Algebra and Advanced Calculus Examination

1. Define the following terms.
  - (a)  $v_1, \dots, v_k$  are linearly independent
  - (b) unitary matrix
  - (c) the adjoint  $L^*$  of a linear transformation  $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$
  - (d) basis for a vector space
  - (e) the orthogonal complement  $S^\perp$  of a subspace  $S$  of  $\mathbb{R}^n$
2.
  - (a) State the version of the Spectral Theorem for self adjoint linear transformations.
  - (b) State a version of the Spectral Theorem for symmetric matrices.
  - (c) Explain how the trace and determinant are related to the characteristic polynomial of a matrix.
  - (d) Explain how one associates a matrix  $A$  to a linear transformation  $L : V \rightarrow V$  given a basis  $v_1, \dots, v_n$  on  $V$ . How does  $A$  change if we take a different basis of  $V$ ?
3.
  - (a) Suppose that  $f(x, y)$  is a function of 2 variables whose first order partial derivatives vanish at the point  $P$ . If the Hessian matrix of second order partial derivatives at  $P$  has eigenvalues 2 and 4, determine whether  $f$  has a local maximum, local minimum, or saddle point at  $P$ .
  - (b) Suppose that  $S$  is a subspace of  $\mathbb{R}^5$  of dimension 2. What is the dimension of the orthogonal complement of  $S$ ?
  - (c) What can one say about the eigenvalues of an orthogonal matrix?
  - (d) What can one say about the eigenvectors of an orthogonal matrix corresponding to distinct eigenvalues?
  - (e) Suppose  $S, T$  are subspaces of  $\mathbb{R}^n$  so that any vector in  $\mathbb{R}^n$  can be written as  $s + t, s \in S, t \in T$ . If  $\dim S = 4, \dim T = 3$ , then what is  $\dim S \cap T$ ?
  - (f) Suppose that  $P$  is a projection matrix which projects  $\mathbb{R}^3$  onto a plane. What are the eigenvalues of  $P$  and the dimensions of the corresponding eigenspaces?
4. Suppose that  $L$  is a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  which reflects each vector  $v$  in the plane  $x + y + z = 0$ .
  - (a) Find an orthonormal basis  $v_1, v_2, v_3$  of  $\mathbb{R}^3$  so that  $v_1, v_2$  is a basis of the plane.
  - (b) Find the matrix of  $L$  with respect to this basis.
  - (c) Find the matrix  $A$  so that  $L(v) = Av$ , where  $v$  is regarded as a column vector on the right hand side.
5. Determine whether each of the following is true or false. No justification is necessary.
  - (a) If  $A$  is a 3 by 4 matrix of rank 2, then  $Ax = b$  has infinitely many solutions for any 3 by 1 column vector  $b$ .
  - (b) If  $A$  is a 3 by 3 matrix with eigenvalues 1,2,3, then  $A$  is diagonalizable.
  - (c) If  $A$  is a Hermitian matrix with real entries, then  $A$  is symmetric.
  - (d) There is a matrix  $A$  so that  $Ax = b$  has exactly 2 solutions.

- (e) If  $\det A = 3$ , then  $A$  is invertible.
- (f) The eigenvalues of an orthogonal matrix must be real numbers.
- (g) If  $A$  is a 3 by 3 matrix with eigenvalues  $0, -1, 3$  and eigenvectors  $v_1 = (1, 1, 0), v_2 = (1, -1, 0), v_3 = (0, 0, 1)$ , then  $A$  is a symmetric matrix.
- (h) If  $A = BC$  is invertible, then the columns of  $C$  are linearly independent.
- (i) If 5 distinct vectors span a vector space  $S$ , then the dimension of  $S$  is 5.
- (j) If a symmetric matrix  $A$  has positive eigenvalues and  $A - 3I$  has a negative eigenvalue, then  $A$  has an eigenvalue between 0 and 3.
6. Find the absolute maximum and absolute minimum of  $f(x, y) = xy^2 + x^2$  on the set  $U = \{(x, y) : x^2 + y^2 \leq 1\}$ .

7. (a) State the Implicit Function Theorem.

(b) Show that the equation  $x \sin(x + y + z) = 0$  determines  $x$  as a function  $\psi(y, z)$ , locally, near the point  $(2, -1, -1)$ . Find  $\psi(-1, -1), \psi_y(-1, -1), \psi_z(-1, -1)$ .

8. (a) Suppose  $P_0 = (x_0, y_0, z_0)$  lies on the level surface  $S = \{P : f(P) = c\}$  and  $g(P_0) \geq g(P)$  for all  $P \in S$ . Assuming that  $f$  and  $g$  have continuous derivatives and  $\nabla f(P_0) \neq 0$ , prove that there is a "Lagrange multiplier"  $\lambda$  such that  $\nabla g(P_0) = \lambda \nabla f(P_0)$ . Hint: Since any normal at  $P_0$  is characterized up to a nonzero constant by being perpendicular to every tangent vector it suffices to show that if  $\gamma(t)$  is a curve in  $S$  with  $\gamma(0) = P_0$ , then  $\nabla g(P_0) \cdot \gamma'(0) = 0$ .

(b) Find the minimum of  $g(x, y, z) = x^2 + y^2 + z^2$  on the surface defined by  $xyz = 1$ .

9. State the Change of Variables Formula for an integral over a region  $R \subset \mathbb{R}^2$  relating  $\int \int_R f dS$  and  $\int \int_{R'} g dS$  where  $\Phi := (\phi, \psi) : R' \rightarrow R$ .

(b) Use the change of variables formula to evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \exp(x^2 + y^2) dx dy$$

(Hint: First sketch the region and then change variables to simplify the integral.)

10. (a) Define  $\int_\gamma F$  where  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  is a differentiable curve, and  $F$  is a vector field.

(b) Show that if  $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ , then the normal to the curve  $\gamma$  obtained by rotating the tangent vector clockwise by  $\pi/2$  is  $(\gamma_2', -\gamma_1')$ . Use this to show that if  $F = (M, N)$ , then

$$\int_\gamma F = \int_\gamma -N dx + M dy.$$

This integral represents the outward flow of the vector field across the curve.

(c) State Green's theorem.

(d) Use Green's theorem to show that the outward flow across the boundary of a region  $R$  is given by the integral over  $R$  of the divergence  $M_x + N_y$ .

(e) Compute this flow by both the line integral over the boundary and the double integral over the region when  $R$  is the region  $\{(x, y) : x^2 + y^2 \leq 1\}$  and  $F = (x, y)$ .