Linear Algebra Vector Calculus Exam 8/26/92

SHOW ALL OF YOUR WORK

1.(i) Define what it means for a complex number, λ, to be an <u>eigenvalue of multiplicity k</u> of a linear transformation, T, of a finite dimensional vector space, V, to itself.
(ii) If 0 is an eigenvalue of T, show that the determinant of T equals 0.
(iii) If V has dimension n and 0 is an eigenvalue of T of multiplicity k=n-r, give an interpretation of r.

2. Give examples of linear transformations $T:V\to W$ for which the number of solutions, v, of the equation Tv=w is

(i) 0 or 1, depending on w
(ii) ∞, independent of w
(iii) 0 or ∞, depending on w
(iv) 1, independent of w.

3. Give reasons if true and a counterexample if false:
(all matrices referred to below are nxn, $i=\sqrt{-1}$ and I is the identity matrix)

(i) If A is Hermitian then A + iI is invertible
(ii) If B is orthogonal then $B + \frac{1}{2}I$ is invertible.
(iii) If C is real then C + iI is invertible
(iv) Every invertible matrix can be diagonalized
(v) Every diagonalizible is invertible
(vi) If D is symmetric positive definite and R is orthogonal then $Q=R^tDR$ has the same eigenvalues as D
(vii) With the same hypotheses as (vi) Q is symmetric positive definite.

4. For a subspace, W, of an inner product space, V, let $W^\perp$ denote the orthogonal complement of W in V. Express $(W_1 + W_2)^\perp$ in terms of $W_1^\perp$ and $W_2^\perp$.

5. Let $S=\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ be a 2x2 real positive definite matrix and let $f(x,y) = ax^2 + 2bxy + cy^2$. Express the maximum and minimum distance from the origin to the graph of the equation f(x,y)=1 in terms of the matrix S.

6. Let $f(x,y):\mathbb{R}^2 \to \mathbb{R}^2$ be the transformation given as

$$f(x,y) = (e^x \cos y, e^x \sin y).$$

Stating any theorem you use, show that f is locally 1-1. Is f globally 1-1? Explain your answer.

7. Find the Jacobian determinant of the transformation from rectangular coordinates (x,y,z) to spherical coordinates (r,θ,ϕ).

8. If $\gamma(t)$ is a simple closed plane curve enclosing a region D and **F** is the vector field $(-y,x)$, show that $\frac{1}{2}\int_{\gamma} \mathbf{F} = \text{area}(D)$.