Linear Algebra and Vector Calculus Exam

January 1992

1. Find the volume in $\mathbf{R}^3$ enclosed by the 2-sphere of unit radius $x^2+y^2+z^2=1$.

2. Let $\mathcal{R}$ be an open set in $\mathbf{R}^3$ and think of $\mathcal{R}$ as some region of physical space filled with air and suppose that $f(x)=$ temperature at $x \in \mathcal{R}$. A gnat is flying through $\mathcal{R}$ so that at time t his position is $x(t)$, and we can assume that x is a differentiable function of time t. At a certain instant t_0, the position of the gnat is $x(t_0)=(3,-1,2)$ and the velocity at that instant is $v(t_0)=x'(t_0)=(5,2,3)$. If $\nabla f(3,-1,2)=(4,2,-1)$ and $\nabla f(5,2,3)=(3,2,5)$, what is the rate of change of temperature experienced by the gnat at time t_0?

3. Calculate the following line integrals.

   (a) $\int_C \vec{F}\cdot d\vec{r}$ where C is the counterclockwise oriented unit circle about the origin in $\mathbf{R}^2$ and $\vec{F}(x,y)=(e^x \sin y)(\vec{i}+\vec{j})$.

   (b) $\int_C \vec{F}\cdot d\vec{r}$ where C is the same as in (a) but now we view $\mathbf{R}^2 \subset \mathbf{R}^3$ as the span of $\{\vec{i},\vec{j}\}$ and $\vec{F}(\vec{r})=\vec{k}\times\vec{r}$ where $\vec{i},\vec{j},\vec{k}$ is the standard right hand frame for $\mathbf{R}^3$.

   (c) $\int_C \vec{F}\cdot dr$ where C is the same as in (b) and $\vec{F}(\vec{r})=\frac{\vec{r}}{\|\vec{r}\|^3}$.

4. Let $A=\begin{bmatrix} \frac{13}{4} & -\frac{3}{4}\sqrt{3} \\ -\frac{3}{4}\sqrt{3} & \frac{7}{4} \end{bmatrix}$.

   (a) Show that the set $\{\mathbf{x}=\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbf{R}^2 \mid \mathbf{x}^t A\mathbf{x}=1\}$ is an ellipse.

   (b) Find the axes of the ellipse.

   (c) Find a rotation of $\mathbf{R}^2$ which transforms the ellipse to the standard form $a\xi^2+b\eta^2=1$, i.e. find a rotation matrix R, such that if

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = R\begin{bmatrix} x \\ y \end{bmatrix}$$

   then $\mathbf{x}^t A\mathbf{x}=a\xi^2+b\eta^2$.

5. Find a basis for the null space of the linear transformation $T:\mathbf{R}^4 \to \mathbf{R}^3$ whose matrix is

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 4 & 7 & 5 \\ 1 & 2 & 4 & 3 \end{bmatrix}.$$

6. Let H be a Euclidean space with inner product $\langle|\rangle_H$. Let L denote the vector space of linear transformations on H. For $x \in L$, the adjoint of x is denoted x^* so $\langle x^*u|v\rangle_H = \langle u|xv\rangle_H$ with $u, v \in H$. Let V be the vector subspace of L consisting of self-adjoint linear transformations. By a frame for H we mean an ordered basis for H, say $(v_1, v_2, \ldots, v_n)$ where $\{v_1, v_2, \ldots, v_n\}$ is a basis for H. If $b = (v_1, \ldots, v_n)$ is a frame for H define

$$\text{trace}_b : L \to \mathrm{R} \quad \text{by} \quad \text{trace}_b(x) = \sum_i x_{ii}$$

where (x_{ij}) is the matrix of $x \in L$ with respect to the frame b. In working any one of the following parts, you may apply the previous parts whether or not they have been worked.

(a) Show that trace_b is linear and that for all $x, y \in L$, $\text{trace}_b(xy) = \text{trace}_b(yx)$ and $\text{trace}_b(x^*) = \text{trace}_b(x)$.

(b) Show that trace_b is independent of the choice of frame b. Define $\text{trace}(x) = \text{trace}_b(x)$ for some frame b for H.

(c) Define $\langle x|y\rangle_L = \text{trace}(y^*x)$ for $x, y \in L$. Show that $\langle|\rangle_L$ is an inner product on H.

(d) If $x \in L$ has matrix (x_{ij}) relative to some frame for H compute $\langle x|x\rangle_L$ in terms of these matrix entries. $\langle x|x\rangle = \cancel{\text{tr}}\; x^*x = \sum_{i=1} x^*_{ij} x_{ij} = \sum_{i=1} |x_{ij}|^2$

(e) Define $p : L \to L$ by $p(x) = \frac{1}{2}(x + x^*)$. Show that p is linear, $p = p^2$, $p = p^*$, and $p(L) = V$. What is the null space of p?
$x^* = -x$

(f) Define the function $h(x) = \text{trace}(x)$ on L. Calculate the gradient $\nabla h(x)$ for $x \in L$, and express this result in terms of $\langle|\rangle_L$. $\nabla h = \frac{\partial h}{\partial x_1} + \cdots + \frac{\partial h}{\partial x_n}$

7. Let H be a Euclidean space with inner product $\langle|\rangle$ so that $\langle x|y\rangle$ denotes the inner product of vectors x, y in H. Let $A : H \to H$ be a symmetric linear tranasformation $A^T = A$ and define $f : H \to \mathrm{R}$ by $f(x) = \langle Ax|x\rangle$. Suppose that λ denotes the maximum value of f on the unit ball $B = \{x \in H : \langle x|x\rangle \leq 1\}$. Show that λ is an eigenvalue of A.

Rayleigh Quotient $\lambda_{max} = \max_{\langle x|x\rangle = 1} \langle Ax|x\rangle$

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$$\lambda_k = \min_{C\,[k-1,n]} \max_{\substack{\|x\|=1 \\ Cx=0}} \langle Ax, x\rangle$$