LINEAR ALGEBRA EXAM

January, 1993

1. True or false: give reason if true, counterexamples if false

   (a) If $\{v_j\}_{j=1}^n$ are eigenvectors corresponding to distinct eigenvalues of a matrix, A, then $\{v_j\}$ are linearly independent.

   (b) If vectors $v_1, \ldots, v_n$ span a finite dimensional vector space, V, then $\dim V = n$.

   (c) If v and w are orthogonal vectors in V and $P : V \to V$ is a projection then Pv and Pw are orthogonal.

   (d) If the $n \times n$ matrix A has rank r, then A^2 also has rank r.

2. Let V be the vector space of polynomials of degree at most 2 with complex coefficients

   (a) What is the dimension of V as a vector space over $\mathbf{C}$?

   (b) Write a basis for V.

   (c) If $D : V \to V$ is the linear transformation which assigns to each polynomial its derivative, find the matrix of D with respect to the basis in part (b).

   (d) What are the eigenvalues of D?

3. (a) Find a real 2×2 matrix A such that $A + xI$ is invertible for all real numbers x.

   (b) Show that there is no 2×2 complex matrix B with $B + zI$ invertible for all complex numbers z.

4. Find a basis for the null-space of the matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$$

5. Let γ be the curve of intersection of the surfaces

$$x + y + z = 0$$

   with

$$x^2 + 2y^2 + 3z^2 = 1.$$

   At which points of γ can you solve uniquely for y and z as functions of x?

6. (a) If D is a region in the plane bounded by the simple closed curve γ, show that the area of D can be computed by evaluating the line integral.

$$\int_{\gamma} x\, dy.$$

(b) Use part (a) to find the area bounded by the curve $x(t) = a\cos t,\ y(t) = a\sin^3 t,\ 0 \leq t \leq 2\pi$.

7. Consider the surface defined by the equation

$$z = 2 - x^2 - y$$

for (x, y) in $\mathbb{R}^2$.

(a) Find the equation of the tangent plane to the surface at $(x, y, z) = (0, 1, 1)$.

(b) Find the normal to the surface at the point $(0, 1, 1)$.

(c) Write down (but do not evaluate) a double integral which evaluates the area of that part of the surface which lies over the region in the first quadrant bounded by $x = 0,\ y = 1$ and $y = x$.