Linear Algebra and Advanced Calculus Written Examination

Thursday, August 25, 1994 – 9:00 A.M. - 1:00 P.M.

Gibson Hall, Room 415

1a. Let A and B be $n \times n$ matrices. Show AB is invertible if and only if BA is invertible.

1b. Let A and B be $n \times n$ matrices. Show that AB and BA have the same eigenvalues.

2. $\{1, x, x^2\}$ is a basis for the vector space of quadratic polynomials. From this basis construct an orthonormal basis with respect to the inner product $(p(x), q(x)) = \int_{-1}^{1} p(x)q(x)\,dx$.

3. Let A be an $n \times n$ matrix with real entries. Prove that $A = 0$ if and only if $\text{trace}(A^T A) = 0$.

4. Let A be a 5×5 complex matrix with characteristic polynomial $f = (x-2)^3(x+7)^2$ and minimal polynomial $p = (x-2)^2(x+7)$. What is the Jordan form for A?

5.a) What can one say about the eigenvalues of an orthogonal matrix?

b) What can one say about the eigenvectors of an orthogonal matrix corresponding to distinct eigenvalues?

c) What can one say about the eigenvalues of a Hermitian matrix?

d) What are the possible eigenvalues of a projection matrix?

e) What does it mean for a matrix to be diagonalizable?

f) What can one say about the eigenvalues of a singular matrix?

(over)

6. Find the absolute maximum and absolute minimum of $f(x,y) = e^{x^2-y^2}$ on R where R is the ring bounded by circles $x^2+y^2 = \frac{1}{2}$ and $x^2+y^2 = 2$ (including the circles).

7. Suppose that $xy^2+ye^{x+y} = 0$. Can x be expressed in terms of y in a neighborhood of $(0,0)$? $(-1,1)$? Why or why not?

8. Let T be a sphere and S its surface. Let u and v be functions with appropriate differentiability. Show

$$\int_T u\nabla^2 v\, d\tau = \int_S u\frac{dv}{dn}\, d\sigma - \int_T \nabla u \cdot \nabla v\, d\tau\ .$$

9. If $f(x)$ is a smooth function in R and if $\lim_{x\to\infty} f'(x) = 0$ prove that $\lim_{x\to\infty} f(x) = c$ where c is a constant.

10. Let $\vec{F} = \frac{y}{x^2+y^2}\vec{i} - \frac{x}{x^2+y^2}\vec{j}$.
Find the line integral of the tangential component of $\vec{F}$ from $(-1,0)$ to $(1,0)$ along the semicircle $y = \sqrt{1-x^2}$.