Linear Algebra-Vector Calculus Exam
January 13, 1997

(1) Suppose that $f(x,y,z)$ and $g(x,y,z)$ are two differentiable functions with the property that $\nabla f \neq 0$ on the surface $S_1 = \{f(x,y,,z) = 0\}$ and $\nabla g \neq 0$ on the surface $S_2 = \{g(x,y,z) = 0\}$.
a) Under what conditions on f and g and their derivatives will $S_1 \cap S_2$ be a differentiable curve?
b) Under the conditions of part a) compute the tangent vector to the curve of intersection, $S_1 \cap S_2$.

(2) Let

$$E = \{(x,y) \in \mathbb{R}^2 \mid x^2 + \frac{y^2}{3} \leq 1\}.$$

Evaluate $\int\int_E f(x,y)\,dA$ where $f(x,y) = \cos(3x^2 + y^2)$.

(3) Find the point(s) on the intersection of the plane $x - y = 1$ and the hyperbolic cylinder $y^2 - z^2 = 1$ which is(are) closest to the origin in $\mathbb{R}^3$.

(4) a)If E is a region in $\mathbb{R}^2$ with a differentiable boundary, ∂E, oriented in the counterclockwise direction, prove that

$$\text{Area}(E) = \frac{1}{2}\int_{\partial E} x\,dy - y\,dx.$$

b) Find the area of the area enclosed by the curve

$$\phi(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right).$$

(5) Let A be an $m \times n$ matrix and B an $n \times r$ matrix. Prove:
a) The nullspace of AB is contained in the nullspace of B.
b) The column space of AB is contained in the collumn space of A.
c) The row space of AB is contained in the row space of B.

(6) Let e_1, e_2 denote the usual basis of $\mathbb{R}^2$.
a) Write the matrix, with respect to the basis $\{e_1, e_2\}$ of $\mathbb{R}^2$, of the linear transformation which rotates a vector in $\mathbb{R}^2$ counter clockwise by $\frac{\pi}{4}$.
b) Write the matrix, with respect to the basis $\{3e_1 - e_2, -3e_1 + e_2\}$ of $\mathbb{R}^2$, of the linear transformation which rotates a vector in $\mathbb{R}^2$ counter clockwise by $\frac{\pi}{4}$.

(7) Let $w = (0,3,0)$. Find the orthogonal projection of w onto the plane spanned by the orthonormal vectors

$$v_1 = \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$$

and

$$v_2 = \left(\frac{-1}{3}, \frac{2}{3}, \frac{2}{3}\right).$$

(8) Consider the matrix, A

$$\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$$

a) Find the eigenvalues and eigenvectors of A.
b) Find an invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D.$$

(9) Show that the matrix

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

cannot be diagonalized.