

LINEAR ALGEBRA-VECTOR CALCULUS EXAM  
JANUARY 9, 2001

1. Define the following terms:
  - a. vector space
  - b. dimension of a vector space
  - c. linear independence of a set of vectors  $\{v_1, \dots, v_n\}$  in a vector space
  - d. eigenvalue of a linear transformation from a vector space to itself
  - e. rank of a linear transformation
  - f. kernel (or nullspace) of a linear transformation
  - g. positive definite quadratic form
  - h. unitary transformation between complex vector spaces

2. For each of the matrices given below give a geometric description of the linear transformation it represents (with respect to the standard basis of the appropriate euclidean space).

a)

$$A = \begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

b)

$$B = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$$

c)

$$C = \begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3.
  - a) Find the eigenvalues of each of the matrices in the previous problem.
  - b) Which of these matrices can be diagonalized over the reals? Explain your answer.
  - c) Show that a real symmetric matrix has real eigenvalues.
  - d) Show that eigenvectors corresponding to distinct eigenvalues of an orthogonal transformation are orthogonal.
4. Find a basis for the nullspace of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 3 & 4 & 5 & 2 & 6 \\ 5 & 8 & 8 & 3 & 10 \end{pmatrix}$$

- b) Compute  $\det(A)$ . State any properties of determinants that you use.

5. In euclidean space  $\mathbb{R}^n$  for  $n = 2, 3$  consider the function

$$r(x) = \sqrt{(x_1^2 + \dots + x_n^2)}.$$

For which values of  $\alpha > 0$  is  $r^{-\alpha}$  integrable on the set  $\{x \mid 0 < r(x) < 1\}$ ? Prove your answer.

6. Find a function  $f(x, y)$  defined and smooth on the set  $\mathbb{R}^2 \setminus (0, 0)$  such that

$$\nabla f(x, y) = \frac{-y}{x^2 + y^2} \mathbf{i} + \frac{x}{x^2 + y^2} \mathbf{j}.$$

7. a) Define what it means for a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  to be differentiable at a point  $(x_0, y_0)$ .

b) For which values of the positive real number  $\alpha$  is the function

$$f(x, y) = \begin{cases} \frac{xy}{(x^2+y^2)^\alpha} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

differentiable at the point  $(0, 0)$ ? Prove your answer.

8. a. Define the divergence of a vector field and state the Divergence Theorem.

b. Suppose that  $D$  is a bounded domain in the plane with a smooth boundary  $C$  and that  $f$  and  $g$  are infinitely differentiable functions in the plane. Use either Green's Theorem or the Divergence theorem to prove that

$$\iint_D f \Delta g \, dA = \oint_C f \nabla(g) \cdot \mathbf{n} \, ds - \iint_D \nabla(f) \cdot \nabla(g) \, dA.$$

where  $\Delta$  is the Laplace operator  $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$  and  $\mathbf{n}$  is the unit outward pointing normal to  $C$ .

9. Suppose that  $(a_{ij})$  is a real symmetric matrix. What are the largest and smallest values of the function

$$\sum_{j=1}^n \sum_{i=1}^n a_{ij} x_i x_j$$

on the set

$$\sum_{k=1}^n x_k^2 = 1?$$

Express your answer in terms of the matrix  $(a_{ij})$ .