LINEAR ALGEBRA/VECTOR CALCULUS EXAM
SPRING 2002

1. Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(\vec{u}, \vec{v}) = 2u_1v_1 - 2u_1v_2 - 2u_2v_1 + 3u_2v_2$

   (a) show that f is an inner product.

   (b) given $\vec{v} = (1, 2)$, use this inner product to find the equation of the line perpendicular to $\vec{v}$ through the point $(2, 1)$.

2. Let A be an $n \times n$ matrix with n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and define the matrix $B = A^2 - 2A + I$. Describe the region in $\mathbb{C}$ where the λ_k's must lie so that the eigenvalues of B are all real and negative.

3. Find the projection of the vector $\vec{v} = (3, 1, 1)$ onto the plane tangent to the surface $(x-2)^2 + (y-3)^2 + (z-6)^2 = 7^2$ at the origin.

4. Let A be an $n \times n$ symmetric matrix with real entries. Suppose that A has a single eigenvalue with multiplicity n. Show that A must be a multiple of the identity.

5. Let

$$A = \begin{pmatrix} 3 & 1 & 199 \\ -1 & 0 & -37 \\ 0 & 0 & 1 \end{pmatrix}$$

   and consider the linear transformation $T(\vec{u}) = A\vec{u}$, for $\vec{u} \in \mathbb{R}^3$. Show that the inverse transformation can be written as $T^{-1} = A^2 - 4A + 4I$.

6. Let

$$f(x, y) = \frac{x^2(1+y) + y^2(1-x)}{x^2 + y^2}$$

   for $x^2 + y^2 \neq 0$. How should $f(0, 0)$ be defined so that f is continuous?

7. Find the minimum distance from the point $(1, 0, 3)$ to the cone $z^2 = x^2 + \frac{1}{4}y^2$ and find the point on the cone where this minimum occurs.

8. Find all the critical points of the function $f(x, y) = (x^2 + 2y^2)e^{-x^2}$ and determine if each critical point is a relative maximum, relative minimum, saddle point, or if your test is inconclusive.

9. For what value of A is the integral

$$\int_C e^x \cos y \; dy - Ae^x \sin y \; dx$$

equal to zero for any smooth closed curve C in the plane?

10. Consider the transformation $u = e^{x^2-y^2}$ and $v = \sin(xy)$.

(a) show that the gradient of u and the gradient of v are orthogonal.

(b) find the Jacobian of this transformation.

(c) are there two continuous functions F and G such that $x = F(u, v)$ and $y = G(u, v)$ in a neighborhood of the point $(u, v) = (e^{-1}, 0)$?