LINEAR ALGEBRA/VECTOR CALCULUS EXAM
SPRING 2004

1. (a) Give a general definition of *inner product* of vectors in $\mathbb{R}^n$.

   (b) Let

$$A = \begin{pmatrix} 7 & 5 \\ 5 & 7 \end{pmatrix}$$

   and define $\langle \vec{u}, \vec{v} \rangle = \vec{u}^T A \vec{v}$ for vectors in $\mathbb{R}^2$. Show that this is an inner product or explain why it is not.

2. Let $\mathcal{P}_2$ be the vector space of all polynomials of degree less than or equal to 2 with real coefficients. Given the inner product $\langle f, g \rangle = \int_0^2 f(x)g(x)dx$, find an orthogonal basis for $\mathcal{P}_2$.

3. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that projects a vector in $\mathbb{R}^3$ onto the plane $x+y-z=0$. Find the matrix A such that $L\vec{v} = A\vec{v}$ with respect to the standard basis.

4. Given the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 3 & -1 \end{pmatrix}$$

   (a) Find a basis for the null space of A.

   (b) Find a basis for the range of A.

   (c) Find a basis for the null space of A^T.

   (d) Find a basis for the range of A^T.

5. Consider the matrix

$$A = \begin{pmatrix} 2 & -3 \\ -3 & 0 \end{pmatrix}$$

   (a) Find the eigenvalues and eigenvectors of A.

   (b) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x^3 + x + ae^{-2y}\ ,\ 2y - \sin(xy) - x)$.

   (a) Linearize $f(x, y)$ about the point $(0, 0)$.

   (b) For what values of a does f have a differentiable inverse in the neighborhood of the point $f(0, 0)$?

7. Let

$$f(x,y) = \begin{cases} \frac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2}, & \text{for } x^2+y^2 \neq 0 \\ c, & \text{for } x^2+y^2 = 0 \end{cases}$$

   Find the value of c that makes $f(x, y)$ continuous or show that there is no such value.

8. Find the shortest distance from the plane $x+2y+z = -6$ to the surface $x^2-4y+z^2 = 1$.

9. Evaluate the integral

$$\int_C (x^2 + 2y^2)\ dy$$

   counterclockwise where C is the circle $(x - 2)^2 + y^2 = 1$.

10. (a) If R is a region in the plane bounded by the simple closed curve C, show that the area of R can be computed with the formula

$$\text{Area of } R \ = -\int_C y\ dx.$$

   (b) Use this result to find the area of the ellipse $x = a\cos t$, $y = b\sin t$, for $0 \leq t \leq 2\pi$.