PRELIMINARY EXAMINATION

LINEAR ALGEBRA AND VECTOR CALCULUS

1. Let C be the closed curve described in the Figure. Evaluate
$$\int_C 2y^3\,dx + (x^4 + 6y^2x)\,dy\,.$$

2. Let C be the intersection of the paraboloid $z = x^2 + y^2$ and the plane $z = y$. Give C its counterclockwise orientation as viewed from the positive z-axis. Evaluate

$$\int_C xy\,dx + x^2\,dy + z^2\,dz\ .$$

3. Find all the local maxima, local minima and saddle points of the function

$$f(x, y) = x^3 - y^3 - 2xy + 6 \ .$$

4. Let $u(x,y) = e^x \cos y$, $v(x,y) = e^x \sin y$. Show that $F(x,y) = (u(x,y), v(x,y))$ is locally one-to-one but is not one-to-one.

5. Let

$$A = \begin{pmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 2 & 0 \\ 1 & -2 & 3 & 3 \end{pmatrix}$$

a) Find a basis for $R(A)$, the range of A.

b) Find a basis for $N(A)$, the null space of A.

b) Find the general solution to $A\underline{x} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$.

6. If $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ is a basis for some vector space V, prove that each $\underline{v}$ in V can be written *uniquely* as a linear combination of these basis vectors.

7. Given an $n \times n$ matrix with a "7" in every entry, find two eigenvalue. Find a corresponding eigenvector for each eigenvalue.

8. (a) An $n \times n$ symmetric matrix A is positive definite if:

(b) Prove that the diagonal entries of a positive definite matrix are positive.

9. For the linear transformation

$$L : \mathcal{P}^2(\mathbb{R}, \mathbb{R}) \rightarrow \mathcal{M}(2,2)$$

$$(\text{polynom. deg} \geq 2) \rightarrow (2 \times 2 \text{ matrices})$$

given by $L(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 + a_2 & a_0 + a_1 \\ a_0 + a_2 & a_0 + a_1 \end{pmatrix}$.

a) Verify that this transformation is *linear*.

b) For the bases

$$\mathcal{V} = \{1, x, x^2\}$$
$$\mathcal{W} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Find the *matrix* that represents the linear transformation with respect to these bases.

c) Find a basis for the nullspace $\mathcal{N}(L)$ and the range $\mathcal{R}(L)$.