PRELIMINARY EXAM IN LINEAR ALGEBRA AND VECTOR CALCULUS

MATHEMATICS DEPARTMENT

1. DIRECTIONS

Print your name in large capital letters at the top of each sheet of paper you turn in. Use at most one problem per sheet and write on one side only. Show all your work and state the major theorems used. Work as many problems as you can in the allotted time.

2. PROBLEMS

1. Suppose that the vector field $\mathbf{F}$ is given by

$$\mathbf{F}(x,y,z) = \frac{(x-1,y,z)}{((x-1)^2+y^2+z^2)^{3/2}} + \frac{(x+1,y,z)}{((x+1)^2+y^2+z^2)^{3/2}},$$

for $x \neq \pm 1$. Let C denote the semicircular path from the origin to the point with coordinates $(0,2,0)$ which passes through the point $(1,1,0)$. Calculate the path integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

2. Suppose that U is an open subset of $\mathbb{R}^3$ whose boundary B is smoothly parametrizable. For $\mathbf{v}$ a point in $\mathbb{R}^3$ let $\mathbf{G_v}$ be the vector field given by

$$\mathbf{G_v}(\mathbf{r}) = \frac{\mathbf{r}-\mathbf{v}}{\|\mathbf{r}-\mathbf{v}\|^3}, \quad \mathbf{r} \neq \mathbf{v}.$$

Define the function $f : \mathbb{R}^3 \setminus B \to \mathbb{R}$ by the surface integral

$$f(\mathbf{v}) = \int_B \mathbf{G_v} \cdot \mathbf{N} dA.$$

Give the numerical values of f and how they depend on $\mathbf{v}$. Explain your answer.

4. Give the equation of the tangent plane at the point $(0,0,0)$ to the surface S in $\mathbb{R}^3$ having equation

$$e^x cos(yz) + e^y cos(xz) = 2.$$

5. Let V be a finite dimensional inner product space and $T : V \to V$ a self-adjoint linear transformation having characteristic polynomial $p_T(z) = (z-2)^3(z-3)^4(z-5)^3$. With $\langle u|v\rangle$ denoting the inner product of u and v in V, find the maximum and minimum values of the function $f(v) = \langle v|T(v)\rangle$ subject to the constraint $\langle v|v\rangle = 1$.

6. Give the characteristic polynomial and the minimal polynomial for the matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

and calculate its determinant and its trace.

7. Suppose that S and T are both linear transformations of the vector space W and the nonzero scalar λ is an eigenvalue of ST. Show that λ is also an eigenvalue of TS.

8. Suppose that I, S, T and X are all linear transformations of the vector space W, that I is the identity and $I - ST$ is invertible. Solve the equation

$$[I + X(I - ST)^{-1}S](I - TS) = I$$

for X and express the inverse of $I - TS$ in terms of I, S, T and $(I - ST)^{-1}$.

9. Suppose that $T : W \to W$ is a linear transformation such that $T^{m+1} = 0$ but $T^m \neq 0$. Suppose that $\{w_1, w_2, w_3, ..., w_p\}$ is a basis for $T^m(W)$ and $T^m(u_k) = w_k$, for $1 \leq k \leq p$. Prove that

$$\{T^i(u_j) : 0 \leq i \leq m,\ 1 \leq j \leq p\}$$

is a linearly independent set.

10. For the linear transformation $A : \mathbb{R}^5 \to \mathbb{R}^4$ having matrix

$$\begin{pmatrix} 1 & 0 & 2 & 4 & 0 \\ 1 & 1 & 5 & 8 & 1 \\ 2 & 1 & 7 & 12 & 1 \\ 0 & 1 & 2 & 3 & 0 \end{pmatrix}$$

find a basis for the null space of A and find a basis for the range of A.

11. Suppose that $T : W \to W$ is a linear transformation of the complex vector space W and λ is an eigenvalue of T. Suppose that $p(z)$ is a complex polynomial. Show that $p(\lambda)$ is an eigenvalue of $p(T)$.

12. For a square matrix A recall the exponential of A denoted e^A is defined by the equation

$$e^A = I + A + (1/2)A^2 + (1/6)A^3 + (1/24)A^4 + ...$$

Suppose that $P_1, P_2, P_3, ..., P_m$ are all square matrices satisfying $P_i P_k = \delta_{i,j} P_i$, where $[\delta_{i,j}]$ is the m by m identity matrix. Suppose that A is in the linear span of $\{P_1, P_2, ..., P_m\}$. Show that e^A is also.

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