Linear Algebra/Vector Calculus Exam

Spring 2007

Name: ____

Soc. Sec. No.: ______________________________________

Signature: ____________________________

Note: no credit will be given if your work is not shown!

1. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$, and C is the unit circle $x^2 + y^2 = 1$ in the plane with the counterclockwise orientation.

2. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$, and C is the curve parametrized by $\vec{r}(t) = (\sin(\pi t), 3^t - 1, 2^t - \cos(\pi t))$, $0 \leq t \leq 1$, from the origin to the point $(0, 2, 3)$.

3. Evaluate the line integral $\int_C (2x + y^2 - 2xe^x)dx + (2xy + 3y + x + \cos y)dy$, where C is the boundary of the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ with the counterclockwise orientation.

4. Evaluate the surface integral $\int\int_S \vec{F} \cdot d\vec{S}$, where $\vec{F} = x^2z\vec{i} - 2xz\vec{j} - (xz^2 + 1)\vec{k}$, and S is the upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$.

5. Let $f : R^2 \to R^2$ be a transformation from the xy-plane to the uv-plane defined by $(u, v) = f(x, y) = (x^2 - y^2, xy)$.

(a). Find the Jacobian for this transformation.

(b). Show that there is an open neighborhood N of the point $(0, 1)$ in the xy-plane such that f has an inverse when restricted onto N.

(c). Show that the gradients of $u = x^2 - y^2$ and $v = xy$ are orthogonal vectors at each and every point in the xy-plane.

(d). Find the linearization of the transformation f about the point $(0, 1)$.

6. Let $f : R^2 \to R$ be a quadratic function defined by $f(\vec{u}) = u_1^2 - 2u_1u_2 + 5u_2^2$, and define $< \vec{u}, \vec{v} >= f(\vec{u} + \vec{v}) - f(\vec{u}) - f(\vec{v})$ for vectors $\vec{u}$ and $\vec{v}$ in R^2.

(a). Show that $< \cdot, \cdot >$ defines an inner product on R^2.

(b). Find a symmetric 2×2 matrix A such that $< \vec{u}, \vec{v} >= \vec{u}^T A \vec{v}$.

7. Let $L : R^3 \to R^3$ be the linear transformation that is defined by reflection about the plane $P : 2x + y - 2z = 0$ in R^3, namely, $L(\vec{u}) = \vec{u}$ if $\vec{u}$ is a vector that lies in the plane P; and $L(\vec{u}) = -\vec{u}$ if $\vec{u}$ is a vector perpendicular to the plane P. Find an orthonormal basis for R^3 and a matrix A such that A is diagonal and A is the matrix representation of L with respect to the orthonormal basis.

8. Let

$$A = \begin{bmatrix} 2 & -2 & 3 & 4 \\ -1 & 1 & 2 & 5 \\ 0 & 0 & -1 & -2 \\ 1 & -1 & 2 & 3 \end{bmatrix}$$

(a). Find a basis for the null space of A.

(b). Find a basis for the range of A.

(c). Find a basis for the null space of A^T.

(d). What is the rank of A.

9. Let

$$B = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$

(a). Use elementary row operations to find the inverse of B.

(b). Use elementary row operations to find the determinant of B.

10. Let $z_1, z_2, \cdots, z_{n+1}$, be $(n+1)$ distinct real numbers. Define a map $F : R^{n+1} \to R^{n+1}$ from the (n+1)-dimensional Euclidean vector space R^{n+1} to itself by $A = F(w)$, where $A = (a_0, a_1, \cdots, a_n)^T$ is the solution of the system $P(z_i) = w_i$, $i = 1, 2, \cdots, n+1$, and $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ is the n-th order polynomial with the components of A as its coefficients. Show that F is a well defined linear transformation from R^{n+1} to itself and it is one-to-one and onto.