Linear Algebra and Vector Calculus Exam Jan. 11, 2008

Solve each problem in the space below its statement. Justify your answers.

1. $\mathbb{R}^3 \xrightarrow{L} \mathbb{R}^3$ is the linear transformation defined by $L(x) = Ax$ for $A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 7 \end{pmatrix}$. Find the null-space of L and the range of L.

2. Let P_2 be the vector space of polynomials of degree 2 or less (with real number coefficients). Define a linear transformation $P_2 \xrightarrow{L} P_2$ by $L(f(t)) \equiv (t-4)f''(t) + (2-t)f'(t) + 2f$
Find the eigenvalues and eigenvectors of L.

3. Let A be a real symmetric n by n matrix. (a) Prove that the eigenvalues of A are real numbers. (b) What can you say about the eigenvectors of A?

4. Let S be the subspace of $\mathbb{R}^3$ defined by $x_1 - x_2 + x_3 = 0$.

a. If $\mathbb{R}^3 \xrightarrow{L} \mathbb{R}^3$ is orthogonal projection onto S, what are the eigenvalues and eigenspaces of L?

b. Find an orthonormal basis for the subspace S.

5. Let $D \equiv \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$ and $f(x,y) \equiv 2x^2+y^2-y$.
Find the global max and min points and the max and min values of f defined on the closed set D.

6. Let S be the sphere in $\mathbb{R}^3$ with center $\underline{o}$ and radius 1, oriented so that the normal vectors point outward.

$S = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$.

Let F be the vector field $F(x,y,z) = (-y, x, z^3)$.

Find the value of the surface integral of F on S,

$$\int_S F = \int_S -y\,dy\,dz + x\,dz\,dx + z^3\,dx\,dy$$

7.a. Let γ be the counterclockwise unit circle in $\mathbb{R}^2$ with center $(0,0)$. Let F be the vector field

$$F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

Calculate the line integral $\int_\gamma F$

b. Prove that there does not exist a differentiable function $f(x,y)$ defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $\nabla f = F$.

8. $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2 \qquad f(x,y) = (x^2 - y^2,\ xe^{2y})$

a. Define what it means for f to have a local inverse [for (x,y) near (a,b)] at $(x,y) = (a,b)$.

b. Show that f has a local inverse at $(1,0)$.

c. If g is the local inverse of f at $(1,0)$, find the differential of g at $(1,1)$.

d. If ϵ, δ are very small, give a good approximation to $g(1+\epsilon,\ 1+\delta)$.