SOLUTIONS

Linear Algebra and Vector Calculus Exam   Jan. 11, 2008

Solve each problem in the space below its statement. Justify your answers.

1. $\mathbb{R}^3 \xrightarrow{L} \mathbb{R}^3$ is the linear transformation defined by $L(x) = Ax$ for $A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 7 \end{pmatrix}$. Find the null-space of L and the range of L.

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 7 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_2 = -x_3$

$x_1 = -3x_3$

$$\boxed{\begin{array}{l} N(L) = \text{span} \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \\ \\ R(L) = \text{span of any two columns of } A \end{array}}$$

2 Let P_2 be the vector space of polynomials of degree 2 or less (with real number coefficients). Define a linear transformation $P_2 \xrightarrow{L} P_2$ by $L(f(t)) \equiv (t-4)f''(t) + (2-t)f'(t) + 2f$
Find the eigenvalues and eigenvectors of L.

Use basis $1 \quad t \quad t^2$ of P_2

$L(1) = 2$

$L(t) = (2-t) + 2t = 2 + t$

$L(t^2) = 2(t-4) + (2-t)2t + 2t^2 = 6t - 8$

Matrix of L wrt basis is

$$A = \begin{pmatrix} 2 & 2 & -8 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

$\boxed{\text{Eigenvalues } 2, 1, 0}$

$E(2) = \text{span } 1$

$\underline{\lambda = 1}$: $\begin{pmatrix} 1 & 2 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & -1 \end{pmatrix} X = 0 \qquad x_3 = 0, \quad x_1 = -2x_2$

$E(1) = \text{span } t - 2$

$\underline{\lambda = 0}$ $\begin{pmatrix} 2 & 2 & -8 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -4 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -10 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{pmatrix}$

$x_2 = -6x_3$
$x_1 = 10x_3$

$E(0) = \text{span } t^2 - 6t + 10$

3 Let A be a real symmetric n by n matrix. ⓐ Prove that the eigenvalues of A are real numbers. ⓑ What can you say about the eigenvectors of A?

Hermitian inner product on $\mathbb{C}^n$

$$\langle z, w\rangle \equiv \sum z_j \bar{w}_j$$

If λ is an eigenvalue of A with eigenvector v,

$$\langle v, Av\rangle = \langle v, \lambda v\rangle = \bar{\lambda}\langle v, v\rangle = \bar{\lambda}\|v\|^2$$

$$\|$$

$$\langle A^* v, v\rangle = \langle Av, v\rangle = \langle \lambda v, v\rangle = \lambda \|v\|^2$$

because A is real and symmetric,

$$A^* \equiv \bar{A}^t = A$$

Therefore $\bar{\lambda} = \lambda$

Eigenvectors of distinct eigenvalues are orthogonal wrt $\langle\, ,\rangle$.

4. Let S be the subspace of $\mathbb{R}^3$ defined by $x_1 - x_2 + x_3 = 0$.

a. If $\mathbb{R}^3 \xrightarrow{L} \mathbb{R}^3$ is orthogonal projection onto S, what are the eigenvalues and eigenspaces of L?

Eigenvalues: 0, 1

eigenspaces: $E(0) = \text{span}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$E(1) = S$

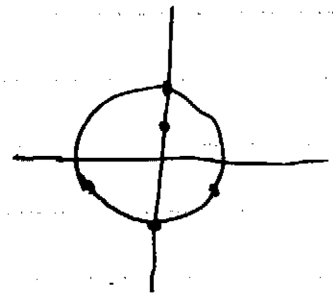
b. Find an orthonormal basis for the subspace S.

Basis $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Apply Gram-Schmidt to get orthonormal basis

$$\boxed{q_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \qquad q_2 = 1/\sqrt{6}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}}$$

5 Let $D \equiv \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}$ and $f(x,y) \equiv 2x^2+y^2-y$.
Find the global max and min points and the max and min values of f defined on the closed set D.



Interior Critical Pts: $f_x = 4x \overset{\text{set}}{=} 0$

$f_y = 2y-1 \overset{\text{set}}{=} 0$

$(x,y) = (0, 1/2)$

Boundary Critical Pts:

$4x = 2x\lambda \longrightarrow x = 0$ or $\lambda = 2$

$2y-1 = 2y\lambda$

if $x = 0$, $y = \pm 1$ $\quad (0,1) \quad (0,-1)$

if $\lambda = 2$, $-1 = 2y$ $\quad y = -1/2 \quad x = \pm\sqrt{3}/2 \quad (\sqrt{3}/2, -1/2) \quad (-\sqrt{3}/2, -1/2)$

$f(0, 1/2) = -1/4$

$f(0,1) = 0$

$f(0,-1) = 2$

$f(\pm\sqrt{3}/2, -1/2) = 2 \cdot \frac{3}{4} + \frac{1}{4} + \frac{1}{2} = 2\frac{1}{4}$

Global max pts	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	max value $2\frac{1}{4}$
	$\left(\frac{-\sqrt{3}}{2}, -\frac{1}{2}\right)$	
Global min pt	$(0, 1/2)$	min value $-1/4$

6. Let S be the sphere in $\mathbb{R}^3$ with center 0 and radius 1, oriented so that the normal vectors point outward:
$S = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$.
Let F be the vector field $F(x,y,z) = (-y, x, z^3)$.
Find the value of the surface integral of F on S,

$$\int_S F = \int_S -y\,dy\,dz + x\,dz\,dx + z^3\,dx\,dy$$

$$= \int_{B(0,1)} 3z^2\,dx\,dy\,dz$$

divergence thm

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 3\rho^4 \cos^2(\phi)\sin(\phi)\,d\rho\,d\phi\,d\theta$$

$$= 2\pi \cdot \frac{3}{5} \quad -\frac{\cos^3(\phi)}{3}\Big|_0^{\pi}$$

$$= 2\pi \cdot \frac{3}{5}\,\frac{2}{3} = \boxed{\frac{4\pi}{5}}$$

7. a. Let γ be the counterclockwise unit circle in $\mathbb{R}^2$ with center $(0,0)$. Let F be the vector field

$$F(x,y) = \left(\frac{-y}{x^2+y^2} , \frac{x}{x^2+y^2} \right)$$

Calculate the line integral $\int_\gamma F$

$$= \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t)) \, dt = \boxed{2\pi}$$

b. Prove that there does not exist a differentiable function $f(x,y)$ defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $\nabla f = F$.

If f existed, $\int_\gamma F = f(\text{end pt } \gamma) - f(\text{begin. pt } \gamma)$
$= 0.$

Therefore no f exists.

8. $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2 \qquad f(x,y) = (x^2-y^2,\ xe^{2y})$

a. Define what it means for f to have a local inverse [for (x,y) near (a,b)] at $(x,y)=(a,b)$.

There is an open set $U \ni (a,b)$ and an open set $V \ni f(a,b)$ and a differentiable map g

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{f} & \mathbb{R}^m \\ \cup & & \cup \\ U & \xleftarrow{g} & V \end{array}$$

such that $g(f(x,y)) = (x,y) \quad \forall (x,y) \in U$

$f(g(u,v)) = (u,v) \quad \forall (u,v) \in V$

b. Show that f has a local inverse at $(1,0)$.

$$df = \begin{pmatrix} 2x & 2y \\ e^{2y} & 2xe^{2y} \end{pmatrix}$$

$$df_{(1,0)} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$ is non-singular so, by the inverse function theorem, a local inverse g exists for f at $(1,0)$

$f(1,0)=(1,1)$ so g is defined on a nbd V of $(1,1)$

c. If g is the local inverse of f at $(1,0)$, find the differential of g at $(1,1)$.

$$dg_{(1,1)} = df^{-1}_{(1,0)} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 \\ -1/4 & 1/2 \end{pmatrix}$$

d. If ϵ, δ are very small, give a good approximation to $g(1+\epsilon, 1+\delta)$.

$$g(1+\epsilon, 1+\delta) \approx g(1,1) + dg_{(1,1)}\begin{pmatrix} \epsilon \\ \delta \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/2 & 0 \\ -1/4 & 1/2 \end{pmatrix}\begin{pmatrix} \epsilon \\ \delta \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \epsilon/2 \\ -\epsilon/4 + \delta/2 \end{pmatrix}$$