

ALGEBRA QUALIFYING EXAM

January 8, 1990, Noon - 4 P.M.

1. State the following definitions:
  - (a) group;
  - (b) subgroup;
  - (c) factor group;
  - (d) normalizer;
  - (e) solvable group;
  - (f) composition series.
2. Prove that a group of order  $p^n$  has a non-trivial center (when  $p$  is prime and  $n > 0$ ).
3. Prove that  $A_n$  is generated by all 3-cycles.
4. Find all groups of order 10. (Answer must be justified.)
5. Give a multiplication table for the group generated by  $a, b$  subject to  $a^4 = b^4 = 1$ ,  $a^2 = b^2$ ,  $ba = a^3b$ .
6. In a commutative ring with identity:
  - (i) state definitions of:
    - (a) ideal;
    - (b) primary ideal;
    - (c) prime ideal;
    - (d) maximal ideal
  - (ii) state all implications between these four concepts;
  - (iii) give examples showing that no other implications exist.
7. Determine the Galois group of  $x^3 - x - 1$  over  $\mathbb{Q}$ .
8. Construct a field with 9 elements.

9. State the definitions of:

(a) injective module; (b) projective module; (c)  $A \otimes_R B$ .

10. Given a pullback of left  $R$ -modules:

$$\begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ g' \uparrow & & \uparrow g \\ \cdot & \xrightarrow{f'} & \cdot \end{array}$$

prove that  $\text{Ker } f' \cong \text{Ker } f$ .

11. Given an abelian group  $A$  describe  $\mathbb{Q} \otimes_{\mathbb{Z}} A$  when: (i)  $A$  is torsion; (ii)  $A$  is torsion-free.

12. Given a short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  of left  $R$ -modules, give conditions which imply that

$$0 \rightarrow D \otimes_R A \rightarrow D \otimes_R B \rightarrow D \otimes_R C \rightarrow 0$$

$\otimes_R A$  is circled,  $\otimes_R B$  is circled, and  $\otimes_R C$  is circled. A handwritten arrow labeled "tensor?" points from the circled  $\otimes_R A$  to the circled  $\otimes_R B$ . Another handwritten arrow labeled "kernel?" points from the circled  $\otimes_R B$  to the circled  $\otimes_R C$ .

is a short exact sequence.

13. State Frobenius' Theorem (for division algebras over  $R$ ).

14. State the definition of adjoint functors.

15. Let  $F$  be a functor from a category  $\mathcal{C}$  to the category of sets, and  $X$  be an object of  $\mathcal{C}$ . Prove the following (Yoneda's Lemma): there is a one-to-one correspondence, which is natural in  $X$  and  $F$ , between the natural transformations from  $\text{Hom}_{\mathcal{C}}(X, -)$  to  $F$ , and the elements of  $F(X)$ .