

ALGEBRA QUALIFYING EXAM

June 6, 1990.

1. State the following definitions:

- (a) group; (b) subgroup; (c) factor group;
(d) normalizer; (e) solvable group; (f) composition series.

2. Prove that a group of order p^n has a non-trivial center (when p is prime and $n > 0$).

3. Prove that two permutations are conjugate if and only if they have the same cycle structure.

4. Find all groups of order 15. (Answer must be justified.)

5. Give a multiplication table for the group generated by a, b subject to $a^4 = b^2 = 1, ba = a^3b$.

6. In a commutative ring with identity:

(i) state definitions of:

(a) ideal; (b) prime ideal; (c) maximal ideal;

(ii) relate these concepts to quotient ring properties.

7. Determine the Galois group of $x^3 + 2x - 1$ over \mathbb{Q} .

8. Construct a field with 8 elements.

9. State the definitions of:

(a) injective module; (b) projective module; (c) $A \otimes_R B$.

10. Given a commutative diagram of left R -modules with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & 0 \\ & & \uparrow f & & \uparrow g & & \uparrow h & & \\ 0 & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & 0 \end{array}$$

in which f and h are surjective, prove that g is surjective.

11. Let A be a free right R -module and B be a free left R -module. Show that $A \otimes_R B$ is a free abelian group.

12. Given a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of left R -modules, state conditions which imply that

$$0 \rightarrow D \otimes_R A \rightarrow D \otimes_R B \rightarrow D \otimes_R C \rightarrow 0$$

is a short exact sequence.

13. State Frobenius' Theorem (for division algebras over R).

14. State the definition of adjoint functors.

15. Let A, B be sets and A^B denote the set of all mappings $B \rightarrow A$. Show that for each set B the evaluation mapping $e_{A,B} : A^B \times B \rightarrow A$ (defined by $e(f,b) = f(b)$) is a natural transformation (of set-valued functors).