

October 1990.

ALGEBRA EXAM

for Master' Degree

1. STATE the axioms of "semigroup" and "group", and PROVE that in a group the equations $ax=b$ and $ya=b$ (a, b given, x, y unknown) are uniquely solvable.

2. STATE the definition of coset, LIST the main properties of cosets, and STATE Lagrange's Theorem and its corollaries.

3. DEFINE "factor group", "homomorphism", "kernel of homomorphism". PROVE that "kernel of homomorphism" and "normal subgroup" mean the same for a subgroup. What are the corresponding concepts for rings?

4. (a) DEFINE "normal chain", "composition chain" and "simple group".
(b) GIVE examples for non-commutative simple groups.
(c) GIVE a composition chain for the quaternion group.

5. (a) STATE the definition of "free groups" in terms of their universal property.
(b) DESCRIBE the elements of a free group generated by a finite number of elements.
(c) EXPLAIN what it means to present a group in terms of generators and relations.

6. STATE the definitions of

(a) ring ; (b) identity; (c) divisor of zero;
(d) domain; (e) division ring; (f) field.

7. (a) DEFINE "prime field" and GIVE a full description of prime fields.
(b) GIVE an example of a finite field which is not a prime field.

8. STATE the four equivalent conditions for a ring R to be *left noetherian*.

9. Let $\alpha: A \rightarrow B$ be a homomorphism between the left R -modules A, B ; R denotes any ring. If X is any left R -module, DESCRIBE the "induced" maps

$$\text{Hom}_R(X, A) \rightarrow \text{Hom}_R(X, B) \quad \text{and} \quad \text{Hom}_R(B, X) \rightarrow \text{Hom}_R(A, X).$$

10. (a) DEFINE *algebraic* and *transcendental* extensions of a field F by an element α .
(b) STATE what $F(\alpha)$ is isomorphic to when α is algebraic or transcendental.

11. Let $f(x) = x^4 - 6x^3 + 4x^2 + 2$ be the polynomial in $\mathbb{Q}[x]$; and let α be any of its roots. Knowing that $f(x)$ is irreducible,

(a) FIND a basis for the field extension $\mathbb{Q}(\alpha)|\mathbb{Q}$;
(b) EXPRESS $1/\alpha^2$ as a linear combination of the basis you found in (a).

12. STATE the definitions of

(a) category; (b) functor; (c) adjoint functor.