ALGEBRA QUALIFYING EXAM

August ~~19~~ 23, 1991

1. (a) State the axioms of a group.

   (b) Prove: if $f : G \longrightarrow H$ is a homomorphism of groups and $T < H$ then $S = f^{-1}T < G$.

2. (a) State the Sylow Theorems.

   (b) Prove that there is no simple group of order 12.

3. State the universal property of a free group and describe its multiplication.

4. (a) Explain what it means that G is the group generated by a and b subject to $R : a^3 = b^4 = 1,\ ba = ab^3$.

   (b) Give a multiplication table for this group G.

5. (a) Define a composition series.

   (b) State the Jordan-Hölder Theorem.

   (c) Give an explicit composition series for S_4.

6. (a) What is a field of quotients?

   (b) Give two distinct examples of non-trivial fields of quotients.

7. Let K be a field.

   (a) What can you say about the rings $K[X]$ and $K[X,Y]$?

   (b) Find three ideals $Q \subset P \subset M$ of $K[X,Y]$ such that M is a maximal ideal, P is prime but not maximal and Q is primary but not prime.

8. Let E be a finite field extension of a field K.

(a) Prove that E is an algebraic extension.

(b) Define the separability degree of E (over K). How does it relate to the degree of E and the characteristic of K?

9. (a) Define the Galois group of $f \in \mathbb{Q}[X]$.

(b) Determine the Galois group of $f(x) = x^3 + x - 2 \in \mathbb{Q}[X]$.

10. Given a commutative diagram of left R-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & 0 \\ & & \| & & \uparrow & (1) & \uparrow & & \\ 0 & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & 0 \end{array}$$

in which the top row is short exact and (1) is a pullback, prove that the bottom row is short exact.

11. (a) Define injective left R-modules.

(b) State the theorem giving all injective abelian groups.

(c) Prove that a direct summand of an injective module is injective.

12. (a) Define $A \otimes_R B$ (where R is a ring).

(b) Find $A \otimes_{\mathbb{Z}} \mathbb{Z}_n$ for any abelian group A and $n > 0$.

13. What can you say about finitely generated modules over principal ideal domains?

14.(a) What is an exterior algebra?

(b) Describe the exterior algebra of the real vector space $\mathbb{R}^3$; give geometric interpretations of its elements and their $\wedge$ products.

15.(a) Define limits in any category.

(b) Give a construction of limits in the category of sets.

16. Prove that right adjoint functors preserve limits.