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BASIC EXAM ALGEBRA

1. (a) STATE the axioms for "semigroup" and "group".
(b) PROVE that in a group the equations ax=b and ya=b (a, b given, x, y unknown) are uniquely solvable.

2. (a) STATE the definition of "coset", and LIST its main properties.
(b) STATE Lagrange's Theorem and its corollaries.
(c) STATE two equivalent definitions of "normal subgroup", and GIVE an example of a subgroup which is not normal.

3. (a) STATE Sylow's theorems.
(b) Consider A_4, the alternating group on 4 letters. Let K be the subgroup which consists of the following permutations:

$$(1), \quad (12)(34), \quad (13)(24), \quad (14)(23).$$

SHOW that this is the only 2-Sylow subgroup in A_4.

✓4. (a) DEFINE "simple group" and GIVE examples for non-commutative simple groups.
(b) PROVE that there is no simple group of order 56.

5. (a) STATE the Homomorphism Theorem and the two Isomorphism Theorems for groups [be careful with the hypotheses].
(b) GIVE an example illustrating one of the cited theorems.

6. (a) DEFINE "normal chain", "composition chain".
(b) SHOW that a normal chain is a composition chain if and only if its factors are simple groups.
(c) GIVE explicitly a composition chain for the quaternion group.

7. (a) STATE the definition of "free groups" in terms of their universal property.
(b) DESCRIBE the elements of a free group generated by a finite number of symbols.
(c) EXPLAIN the meaning of "presention of a group in terms of generators and relations".
✓(d) GIVE a presentation for the quaternion group.

8. STATE the definitions of

(a) ring ;	(b) divisor of zero;	(c) division ring;
(d) field;	(e) field of quotients.	

9. (a) DEFINE "left", "right" and "twosided ideals" of a ring.
(b) GIVE examples for each of them in a 2×2 matrix ring over a field.
✓(c) PROVE that "kernel of homomorphism" and "twosided ideal" mean the same for a subring.

10. (a) DEFINE "prime field".
(b) STATE and PROVE the theorem giving a full description of prime fields.
(c) GIVE an (the only) example of a finite field with exactly 4 elements.

11. (a) STATE the definition of "simple ring".
(b) SKETCH the proof that the full matrix ring over a field F is a simple ring.

12. (a) STATE the four equivalent conditions for a ring R to be "left noetherian".

✓(b) Are the polynomial rings $F[x,y,\ldots,z]$ (F a field) noetherian?

13. (a) DEFINE the "tensor product $A \otimes_R M$" for a right R-module A and a left R-module M.

(b) Let $\alpha: A \to B$ be a homomorphism between the right R-modules A, B. If M is any left R-module, DESCRIBE the "induced" map

$$A \otimes_R M \to B \otimes_R M.$$

✓(c) FORMULATE the exact sequences for the tensor products.

14. (a) STATE the definitions of "free" and "projective" R-modules.

(b) GIVE an example of a projective module that is not free.

✓15. (a) DETERMINE the degree of the splitting field N of the polynomial $f(x) = x^4 - 3$ over the rational number field $\mathbb{Q}$.

(b) FIND the Galois group of f(x).

(c) WHAT are the fields between $\mathbb{Q}$ and N?

16. FORMULATE what you know about the solvability of algebraic equations by radicals. (You need not prove anything, but have to state the relevant definitions and theorems.)

17.(a) STATE the definition of "category", and GIVE several examples for categories.

(b) DEFINE "functor", "adjoint functor".

(c) GIVE examples for the adjoint situation between various categories.