Aug 31, 1992.

BASIC EXAM ALGEBRA

Note: *1. For each problem, STATE the relevant definitions before solving the problem.*
2. Out of the following 17 problems, 15 are to be solved.

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1. (a) STATE the axioms for "group" and " abelian group" .
(b) PROVE that in a group G the equations ax=b and ya=b (a, b given in G) are uniquely solvable for x and y in G.

2. (a) STATE the definition of " coset ", and LIST its main properties.
(b) PROVE that the inverse of a left coset is a right coset, and conclude that left index is equal to right index.
(c) STATE Lagrange's Theorem and its corollaries.
(d) STATE two (equivalent) definitions of "normal subgroup".

3. (a) STATE (but do NOT prove) the three Sylow theorems.
(b) Consider A_4, the alternating group on 4 letters. FIND all 2- and 3-Sylow subgroups in A_4.

4. (a) DEFINE "simple group" and GIVE examples for commutative as well as for non-commutative simple groups.
(b) PROVE that there is no simple group of order 39.

5. (a) DEFINE "normal chain", "composition chain" .
(b) SHOW that a normal chain is a composition chain if and only if its factors are simple groups.
(c) GIVE explicitly a composition chain for S_4, the symmetric group on 4 letters.

6. (a) STATE the definition of "free groups" in terms of their universal property.
(b) DESCRIBE the elements of a free group generated by a finite number of symbols.
(c) EXPLAIN the meaning of "presention of a group in terms of generators and relations".
(d) GIVE a presentation for the quaternion group.

7. STATE the definitions of
   (a) ring ;
   (b) left, right and twosided ideal;
   (c) division ring;
   (d) field;
   (e) field of quotients.

8. PROVE that "kernel of ring homomorphism" and "twosided ideal" are equivalent properties for a subring.

9. (a) STATE the theorems giving a full description of finite fields.

(b) VERIFY the ireducibility of the following polynomial over the prime field of characteristic 2:

$$f(x) = x^3 + x + 1.$$

(c)USE this f(x) to give an explicit example of a finite field with exactly 8 elements.

10.(a) STATE the definition of "simple ring".

(b) SKETCH the proof that the full matrix ring over a field F is a simple ring.

11. (a) STATE the four equivalent conditions for a ring R to be "left noetherian".

(b) Which polynomial rings are noetherian?

12. (a) DEFINE the "tensor product $A \otimes_R B$" for a right R-module A and a left R-module B.

(b) If A and B are finite cyclic groups of orders m and n, respectively, prove that the tensor product $A \otimes_{\mathbb{Z}} B$ is again a finite cyclic group. Determine its order.

13. (a) STATE the definitions of "free" and "projective" R-modules.

(b) PROVE that free modules are projective.

(c) GIVE an example of a projective module that is not free (over a suitable ring R).

14. Let α denote a root of $f(x) = x^4 - 6x^3 + 4x^2 + 2$.

(a) Prove that f(x) is irreducible;

(b) find a basis for the field extension $\mathbb{Q}(\alpha)|\mathbb{Q}$;

(c) express $1/\alpha$ as a linear combination of the basis you found in (b).

15. (a) DETERMINE the degree of the splitting field N of the polynomial $f(x)= x^4 - 7$ over the rational number field $\mathbb{Q}$.

(b) FIND the Galois group of f(x).

(c) WHAT are the fields between $\mathbb{Q}$ and N?

16. TELL what you know about geometric constructions. (You need not prove anything, but have to state the relevant definitions and theorems.)

17.(a) STATE the definition of "category", and GIVE several examples for categories.

(b) DEFINE "functor", "adjoint functor".

(c) What are the left adjoints to the following forgetful functors?

(i) Monoid $\to$ Set;

(ii) $\text{Ring}^1 \to$ Monoid; (forget addition in the category of rings with 1)

(iii) $\text{Mod}_R \to$ Ab;

(iv) Ab $\to$ Group.