August, 1993.

BASIC EXAM ALGEBRA

1. (a) STATE the axioms for "group" and "abelian group" .
(b) PROVE that in a group the equations ax = b and ya = b (a, b given, x, y unknown) are uniquely solvable.

2. (a) PROVE that a subgroup is normal if and only if it is the kernel of a homomorphism of the group.
(c) GIVE an example of a subgroup which is not normal.

3. (a) DEFINE "conjugacy" between elements of a group.
(b) STATE the result on the number of conjugates of an element.

4. DEFINE "simple group" and SHOW that there is no simple group of order 63.

5. (a) DEFINE "normal chain", "composition chain".
(b) PROVE that a normal chain is a composition chain exactly if its factors are simple groups.
(c) GIVE explicitly a composition chain for the symmetric group S_4 on 4 letters.

6. DEFINE "center" of a group G. Do you know of any theorem that states that certain groups have non-trivial center? If so, STATE it.

7. (a) STATE the definition of "free groups" in terms of their universal property.
(b) DESCRIBE the elements of a free group generated by a finite number of symbols.
(c) EXPLAIN the meaning of "presention of a group in terms of generators and relations".
(d) GIVE a presentation for a cyclic group of order n.

8. (a) FORMULATE the fundamental theorem on finite abelian groups.
(b) ILLUSTRATE it on the example of the relatively prime residue classes mod 14.

9. (a) DEFINE "left", "right" and "twosided ideals" of a ring.
(b) GIVE examples for each of them in a 2×2 matrix ring over a field.

10. (a) DEFINE "prime field".
(b) STATE and PROVE the theorem giving a full description of prime fields.
(c) GIVE an (the only) example of a finite field with exactly 4 elements.

11. (a) DEFINE "GCD", "P.I.D." and "Euclidean domain".
(b) DESCRIBE the relation between rings with GCD, PID's and Euclidean domains?

12. (a) STATE equivalent conditions for a ring R to be "left noetherian".
(b) FORMULATE Hilbert's Basis Theorem?

13. DEFINE the "tensor product $A \otimes_R M$" for a right R-module A and a left R-module M.
(b) What can you say about $R \otimes_R M$?

14. (a) STATE the definitions of " free" and "projective" R-modules.
(b) DESCRIBE the connection between free and projective modules.
(c) GIVE an example of a projective module that is not free.

15. (a) DEFINE *algebraic* and *transcendental* extensions of a field K by an element α.

(b) PROVE that $K(\alpha) \cong K[x]/(f(x))$ or $K(x)$ according as α is algebraic or transcendental where $f(x)$ is ... .

16. Let $f(x) = x^4 - 10x^3 + 2$ be a polynomial in $\mathbb{Q}[x]$, and let α be any of its roots. Knowing that $f(x)$ is irreducible,

(a) FIND a basis for the field extension $\mathbb{Q}(\alpha)|\mathbb{Q}$;

(b) EXPRESS $1/\alpha^3$ as a linear combination of the basis you found in (a).

17. (a) DETERMINE the degree of the splitting field N of the polynomial $f(x) = x^4 - 11$ over the rational number field $\mathbb{Q}$.

(b) FIND the Galois group of $f(x)$.

(c) WHAT are the fields between $\mathbb{Q}$ and N?

18.(a) STATE the definition of "category", and GIVE examples for important categories.
(b) DEFINE "functor", "adjoint functor".
(c) GIVE examples for the adjoint situation between various categories.