ALGEBRA QUALIFYING EXAM - Monday, August 29, 1994

1. State the definitions of the following terms: (a) group; (b) subgroup; (c) factor group; (d) normalizer; (e) composition series; (f) Sylow subgroup.

2. State the Sylow Theorems.

3. In a commutative ring R with identity, state the definitions of: (a) an ideal; (b) a prime ideal; (c) a maximal ideal. Briefly state how these concepts relate to the quotient ring (R modulo the ideals).

4. Give three (preferably more) examples of commutative Noetherian rings. (Proofs are not required.)

5. Give one definition for each of: (a) a module; (b) a projective module; (c) an injective module; (d) an initial object [in a category]; (e) a natural transformation; (f) adjoint functors.

6. Give three (preferably more) examples of limits and colimits (in the category theory sense). (Proofs are not required.)

7. Find all integer solutions of $12x - 5y = 3$.

8. Find the group $G = \langle a, b;\ a^2 = b^2 = (ab)^3 = 1 \rangle$.

9. Find all conjugacy classes in A_4.

10. Let G be a finite group and p be a prime number. Prove that the order of G is a power of p if and only if the order of each element of G is a power of p.

11. Let R be a ring with identity. Prove that every [proper] left ideal of R is contained in a maximal left ideal.

12. Find the Galois group of $X^3 - 3X + 1$ over $\mathbb{Q}$.

13. Let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} \\ C & \longrightarrow & D \end{array}$$

be a pullback of modules in which γ is surjective. Prove that α is surjective.

14. Find $\mathbb{Z}_6 \otimes_{\mathbb{Z}} \mathbb{Z}_8$.

15. Let K be a field and V be a vector space over K with basis e_1, e_2. Describe $V \otimes_K V$. Describe the exterior algebra of V.

16. Let $\mathcal{C}$ be a category with small Hom sets, X be an object of $\mathcal{C}$, and $F : \mathcal{C} \longrightarrow \mathit{Sets}$ be a functor. Prove that the natural transformations $\mathrm{Hom}_{\mathcal{C}}(X, -) \longrightarrow F$ constitute a set and that there is bijection

$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(X, -), F) \longrightarrow F(X).$$