

ALGEBRA QUALIFYING EXAM - Thursday, January 5, 1995

1. State the definitions of the following terms: (a) group; (b) factor group; (c) dihedral group; (d) centralizer [of a subgroup].

2. State the Sylow Theorems.

3. (a) Give several equivalent definitions of Noetherian rings (in the commutative case).  
(b) State the Hilbert Basis Theorem.

4. Let  $F$  be a field with four elements. Give addition and multiplication tables for  $F$ .

5. Give one definition for each of: (a) a module; (b) an injective module; (c) the limit of a diagram [in a category]; (d) a cokernel.

6. Give the definition and three examples of adjoint functors. (Proofs are not required.)

7. Find  $\phi(24)$ . ( $\phi$  is Euler's  $\phi$  function.)

8. Find the group  $G = \langle a, b; a^4 = 1, a^2 = b^2, ba = a^3b \rangle$  (list the elements and prove that you have the correct group).

9. Prove that two permutations are conjugate in  $S_n$  if and only if they have the same cycle structure.

10. Let  $G$  be a finite group of order  $p^n$  (where  $p$  is a prime number and  $n \geq 1$ ). Prove that the center of  $G$  is not trivial.

11. Prove that every vector space over a field has a basis.

12. Find the Galois group of  $X^3 + 2X - 1$  over  $\mathbb{Q}$ .

13. Let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow \alpha & & \downarrow \gamma \\ C & \longrightarrow & D \end{array}$$

be a pushout of modules in which  $\alpha$  is injective. Prove that  $\gamma$  is injective.

14. Find the tensor product  $\mathbb{Z}_6 \otimes_{\mathbb{Z}} \mathbb{Z}_9$ .

15. Let  $R$  be a commutative ring with identity and  $F$  be a free  $R$ -module with basis  $\{x, y\}$ . Show that  $F \otimes_R F$  and the exterior algebra of  $F$  are free  $R$ -modules.

16. Prove that a direct sum of (any number of) projective modules is projective.