

ALGEBRA QUALIFYING EXAM — August 26, 1998

1. State the Sylow Theorems; include all relevant definitions.
2. Find all non-isomorphic groups of order 175.
3. Let A and B be subgroups of a finite group G . Show that $|AB| = \frac{|A||B|}{|A \cap B|}$.
4. Find all groups generated by two elements a and b such that $a^4 = b^2 = 1$ and $bab = a^{-1}$.
5. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 5 & 6 & 2 \end{pmatrix}$. How many conjugates does σ have in S_6 ?
6. Find the Galois group of the polynomial $X^3 + X - 3 \in \mathbb{Q}[X]$.
7. Draw the addition and multiplication tables of a field with 4 elements.

In the following problems, all rings have identity elements.

8. Give an example of a UFD which is not a PID.
9. Give an example of a projective module which is not free.
10. Prove the following: if a ring R has only one maximal left ideal L , then L is also the only maximal right ideal.
11. Let R be a ring such that every left R -module is free. Prove that R is a division ring.
12. Let A be a left R -, right S -bimodule, and B be a left R -, right T -bimodule. Explain how $\text{Hom}_R(A, B)$ is a left S -, right T -bimodule.
13. Let M be a left R -module and I be an ideal of R . Show that $(R/I) \otimes_R M \cong M/IM$.
14. Let R be a commutative ring, and F be a free R -module with basis $\{e, f\}$. Give a basis of the exterior algebra of F .
15. Let R be a commutative ring, and F be a free R -module with basis $\{e, f\}$. Give a basis of the symmetric algebra of F .
16. State the Adjoint Functor Theorem.