ALGEBRA QUALIFYING EXAM - May 1999

1. Prove that every subgroup of index 2 is normal.

2. State the Sylow Theorems; include all relevant definitions.

3. Find all non-isomorphic groups of order 22.

4. To which known group is $\langle a, b;\ a^3 = 1,\ bab = a,\ aba = b\rangle$ isomorphic? (find the group, give a multiplication table, and prove that your guess is correct).

5. State the Hilbert Basis Theorem.

6. Give the addition and multiplication tables of a field of order 4.

7. Find the Galois group of the polynomial $X^3 + 2X - 1$ over $\mathbb{Q}$.

8. Give an example of a UFD which is not a PID.

9. Give an example of a projective module which is not free.

10. Prove that every direct sum of projective modules is projective.

11. State the Wedderburn-Artin Theorem (on semisimple artinian rings).

12. Let A be a left R-, right S-bimodule and B be a left R-, right T-bimodule. Explain how $\mathrm{Hom}_R(A, B)$ is a left S-, right T-bimodule.

13. Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of left R-modules and $\gamma : C' \longrightarrow C$ be a homomorphism. Show that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \gamma & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

in which the top row is exact.

14. Let R be a commutative ring and F be a free R-module with basis $\{e, f\}$. Give a basis of the exterior algebra of F.

15. State the Adjoint Functor Theorem; include all relevant definitions.

16. Prove that every monomorphism in the category of left R-modules is injective.