May 27, 2004. Name: ....

Qualifying Exam Algebra

1. State and prove Euler's congruence.

2. Solve the diophantine equation $3x - 6y + 8z = 1$.

3. State the definitions of

(a) group; (b) subgroup; (c) left coset; (d) normal subgroup; ; (e) center; (f) factor group; (g) symmetric and alternating groups.

4. Define 'commutator subgroup' G' of a group G, and show that the factor group G/G' is commutative.

5. Define (a) normal chains, (b) composition chains, (c) solvable group. State the Jordan-Hölder theorem.

6. State the definition of Sylow subgroup, and formulate the three Sylow theorems.

7. State and prove Cayley's theorem.

8. Identify the group that is given by the presentation $G = \langle x, y \mid x^4 = y^2 = e, yxy = x^{-1} \rangle$.

9. State the definition of prime fields, and state the theorems you know about them.

10. Define 'Noetherian ring' by stating and proving three equivalent conditions.

11. Let F be any field and $p(x) \in F[x]$ an irreducible polynomial. Prove that if α is a root of $p(x)$, then $F(\alpha) \cong F[x]/(p(x))$.

12. State the main theorem of Galois theory, and explain how it is applied to the problem of solvability of algebraic equations by radicals. (State the necessary definitions.)

13. Determine the Galois group of the polynomial $f(x) = x^4 - 5$ over the rational field $\mathbb{Q}$.

14. State the universal property of tensor products, and sketch the way their existence can be shown.

15. Define push-outs and identify the push-out module P in the following diagram where A, B are left R-modules:

$$\begin{array}{ccc} 0 & \longrightarrow & B \\ \big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ A & \xrightarrow{\alpha} & P \end{array}$$

Explain your answer.

16. Construct a field with 9 elements, and tell how to compute with the elements.

17. Define:
(a) category; (b) initial and terminal object; (c) equalizer and coequalizer; (d) contravariant functor; (e) natural transformation; (f) equivalence of categories.

18. Define adjoint functors, and as an application find the left adjoints to the following forgetful functors:
(a) $U : \mathcal{GP} \to \mathcal{SET}$; (b) $U : \mathcal{AB} \to \mathcal{GP}$; (c) $U : R\text{-}\mathcal{MOD} \to \mathcal{AB}$.