ALGEBRA QUALIFYING EXAMINATION
May 2005

1. State the Sylow Theorems.

2. State the Hilbert Basis Theorem.

3. State the definition, and one or two additional properties, of adjoint functors.

4. State the Adjoint Functor Theorem.

5. Find all integer solutions of $3x - 2y = 1$.

6. Let A and B be subgroups of a finite group. Show that $|AB| = \dfrac{|A|\,|B|}{|A \cap B|}$.

7. Find the group $G = \langle a, b \mid a^2 = b^2 = (ab)^3 = 1 \rangle$ (list the elements of G, draw a multiplication table, and prove that your guess is correct).

8. Construct a field with four elements; give its addition and multiplication tables.

9. Find the Galois group of $X^3 - 3X + 1 \in \mathbb{Q}[X]$.

In what follows, all rings have an identity element, and all modules are unital.

10. A left ideal is nil when all its elements are nilpotent. Prove that every nil left ideal of a left artinian ring is nilpotent.

11. Give an example of a projective module which is not free.

12. Prove that a direct product of (any number of) injective modules is injective.

13. Prove the following: if P is a projective right R-module, and $A \longrightarrow B$ is injective, then $P \otimes_R A \longrightarrow P \otimes_R B$ is injective.

14. Let R be a commutative ring and F be a free R-module with basis $\{e, f, g\}$. Give a basis of the exterior algebra of F. (No proof is required.)