

Algebra qualifying exam

January 16, 2006.

Please write only your assigned number on the top right corner of your pages. This will ensure anonymity in the grading.

This exam consists of three parts: Groups, Modules and Rings, Fields. Each part has 5 problems. You should return 4 in each part.

Part 1

- (1) a) Define *cyclic group* and *generator* for a cyclic group G .
- b) Let $n \in \mathbb{N}$. Describe the largest subset of integers modulo n that form a group under multiplication. We will denote this group by \mathbb{Z}_n^\times .
- c) Let p be a prime number. Compute the order of $1 + p$ in the group $\mathbb{Z}_{p^n}^\times$.
- (2) Let D_{2n} be the dihedral group of order $2n$, generated by two elements r and s , subject to the relations
- $$r^n = s^2 = 1, rs = sr^{-1}.$$
- Let k be a positive divisor of n .
- a) Prove that the group H generated by r^k is a normal subgroup of D_{2n} .
- b) Prove that D_{2n}/H is isomorphic to D_{2k} .
- (3) Let p be a prime number. Describe all groups of order p^2 .
- (4) Define *solvable group*. Give an example of a solvable group and a nonsolvable one. Explain why they are called solvable, that is, something is being solved.
- (5) Describe all finitely generated abelian groups.

Part 3

- (1) Define the characteristic of a field. Show that any field contains a subfield which is isomorphic to either \mathbb{Q} or \mathbb{F}_p .
- (2) Let α be a root of $x^3 - 3x + 1$. Show that $\sqrt{2}$ is not an element of $\mathbb{Q}(\alpha)$.
- (3) Let F be a field. Define the splitting field of a polynomial $f(x)$ in $F[x]$.
- (4) Is it possible to generate the extension $\mathbb{Q}[\sqrt{2}, \sqrt{5}]$ by adjoining a single element β ? Give an explicit value for β .
- (5) Let F be a finite field of q elements (q is a power of a prime p). Let V be a vector space over F of dimension n . Calculate the order of $GL(V)$. This is the general linear group over V .