ALGEBRA QUALIFYING EXAM

August 28, 2007

This exam consists of three parts: Groups, Rings and Fields. Each part has 5 problems. You should return 4 problems in each part.

PART I

(1) In each of the following situations, is the group G solvable, why?
   (a) $G = S_3 \times S_3 \times S_3$, where S_3 is the symmetric group of degree 3.
   (b) $G = S_5$, the symmetric group of degree 5.
   (c) $|G| = 2007$.

(2) Let D_{2n} be the dihedral group of symmetries of a regular n-gon.
   (a) Exhibit all composition series of D_8. Is D_8 solvable, why?
   (b) Let m be an odd positive integer. Show that $D_{4m} \cong D_{2m} \times Z_2$.

(3) Let A be a nonempty set and let S_A be the permutation group on A. Assume that G is an abelian, transitive subgroup of S_A. Show that $\sigma(a) \neq a$ for all $\sigma \in G \backslash \{1\}$ and all $a \in A$. Deduce that $|G| = |A|$.

(4) Let $p < q$ be two prime numbers and let G be a group of order pq. Suppose $Q \in Syl_q(G)$. Show that $Q \trianglelefteq G$.

(5) State the Fundamental Theorem of finitely generated abelian groups.

PART II

(1) Let R be a commutative ring with identity. Show that a polynomial ring in more than one variable over R is not a Principal Ideal Domain.

(2) Let R be a domain and let m, n be relatively prime positive integers. Prove that the ideal $(x^m - y^n)$ is a prime ideal in $R[x, y]$.

(3) Show that the following polynomials are irreducible in $\mathbb{Z}[x]$:

$$\text{(a) } x^3 - 3x - 3, \qquad \text{(b) } x^3 - 3x - 1.$$

(4) Define Unique Factorization Domains (UFDs). Give an example (with explanation) of a domain which is not a UFD.

(5) Let R be the ring of all continuous functions from the closed interval $[0, 1]$ to $\mathbb{R}$ and for each $c \in [0, 1]$ let $M_c = \{f \in R \mid f(c) = 0\}$. Show that if M is any maximal ideal of R then there is a real number $c \in [0, 1]$ such that $M = M_c$.

PART III

(1) Let F be a finite field of characteristic $p > 0$. Prove that $|F| = p^n$ for some positive integer n.

(2) Define an *algebraic* field extension. Give an example of an *infinite* algebraic field extension (i.e. an algebraic field extension with infinite index).

(3) Let K/F be a finite field extension. Prove that K is a splitting field over F if and only if every irreducible polynomial in $F[x]$ that has a root in K splits completely in $K[x]$.

(4) Is the field extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ a *simple* extension? If so, explain why and give an explicit primitive generator of $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$.

(5) Determine the Galois group of the polynomial $f(x) = x^3 - 3x - 1 \in \mathbb{Q}[x]$.