ALGEBRA BASIC EXAM January 2008

Do ALL problems.

1. Let $H_1, H_2 \subset G$, be subgroups of a group G. STATE , do not prove, two different necessary and sufficient conditions in order that G is the internal direct sum of H_1, H_2. Next state a condition that guarantees that G is the semi -direct product of H_1 and H_2.

2. Let $H \subset G$ be finite groups. Prove that $|H|$ divdes $|G|$, where $|G|$ denotes the cardinality any set such as G.

3. (a) State the three Sylow Theorems. (b) Show that there is no simple group of order 56.

4. Let G be a finite group and and $Z(G)$ the center of G. Suppose that $G/Z(G)$ is isomorphic to a cyclic group of order p where p is a prime, ie $G/Z(G) \cong [\mathbb{Z}/(p)]$. Prove that G is Abelian.

5. Define **composition series** and STATE the Jordan Holder Theorem about composition series of a finite group G.

6. Give a multiplication table for the group generated by $< i, j \mid i^4 = 1, i^2 = j^2, j^{-1}ij = i^{-1} >$.

7. List all possible non isomorphic Abelian group of order $n = 72$ as direct sums of cyclics in such a way that the order of each factor is a divisor of of the order of the preceding one.

8. State but do NOT prove Baer's criterion for the injectivity of a module over any ring. Over the ring of integers $\mathbb{Z}$, PROVE that the rationals $\mathbb{Q}$ form an injective module.

9. Let $F = \mathbb{Z}/(2)$, let K be the splitting field of $x^3 + x + 1 \in F[x]$, and α a root of this polynomial in K. (a) Show that $x^3 + x + 1$ is irreducible in $F[x]$. (b) Find explicitly a vector space basis of K over F in terms of α. (c) Find $1/(\alpha^2+1) \in F[\alpha]$.

10. Let $\mathbb{Q}$ be the rationals, and $\mathbb{Q}(\alpha)$ the splitting field of $x^4 + 1$ over $\mathbb{Q}$, where $\alpha^4 + 1 = 0$ (a) Find the Galois group of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. (b) Exactly how many distinct subfields F are there in $\mathbb{Q} \subsetneq F \subsetneq \mathbb{Q}(\alpha)$, and how many of these are normal extensions of $\mathbb{Q}$?

11 Let $D \subsetneq F$ a commutative integral domain and its field of quotients. (1)State two different necessary and sufficient conditions in order for D to be a unique factorization domain. (2) Consider the following properties which a commutative domain may have : Euclidean, principal ideal domain, unique factorization domain, Noetherian. If D has any one of these properties, do the polynomial rings $D[x]$, $D[x, y]$, $F[x]$, and $F[x, y]$ inherit these properties ?

12. Define F is a **free** module and P is a **projective** (left or right) R-module. Prove that a free module is projective. Show that over the ring of integers $R = \mathbb{Z}$, the rationals modulo the integers $\mathbb{Q}/\mathbb{Z}$, is not projective.

13. Let $S \subset R$ be a multiplicative semigroup not containing 0 of a commutative ring R with identity. Prove that there exists a prime ideal $P \lhd R$ not intersecting S.

14. Define $A \otimes_R B$ for right and left R-modules A and B. Simplify the following., i.e. express them as being isomorphic to something not involving "$- \otimes_R -$": $(R \oplus R) \otimes_R B$, $M(R_n) \otimes_R M(R_m)$ where $M(R_n)$ is the $n \times n$ matrix ring over R.

15. Define **natural isomorphism of functors, adjoint functors**. Give two different examples of adjoint functors. Define categories $\mathcal{C}$ and $\mathcal{D}$ are **equivalent**.

16. Give a vector space basis and a multiplication table for this basis for the exterior algebra of the real vector $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$.