

Written Qualifying Examination for Analysis

August 26, 1991

Instructions: Please return this sheet with your work. All questions carry equal weight and partial credit is given only if substantial progress is made toward successful solution, i.e., a good idea. Some parts of a question may be harder (and hence may be weighed more heavily) than other parts.

This examination consists of three parts. Part A for complex analysis, Part B for real analysis, and Part C for functional analysis.

Question		Points
Part A	1	
	2	
	3	
	4	
Part B	1	
	2	
	3	
	4	
Part C	1	
	2	
Total		

Part A. Complex Analysis

1. Let \mathcal{F} be the family of analytic functions from the unit disk $D = \{z \mid |z| < 1\}$ to itself. Show that $a = \sup \{|f'''(0)| \mid f \in \mathcal{F}\}$ is finite and is actually attained, i.e., there exists $f \in \mathcal{F}$ with $|f'''(0)| = a$. Compute a and exhibit a maximizing function.

2. Let $f(z) = \frac{z}{e^z - e^{-z}}$. We have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

in a neighborhood of $z = 0$. Evaluate a_0 and a_1 . Find the radius of convergence of this power series.

3. Use contour integration to evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{e^{itx}}{x^2 + 1} dx$$

for t real.

4. (a) Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function. Suppose that

$$f(\mathbb{C}) \cap \{z \in \mathbb{C} \mid z = x + iy, x + y = 0, x, y \in \mathbb{R}\} = \emptyset.$$

Prove that f is a constant function.

(b) Let $\Omega \subset \mathbb{C}$ be a domain and let $f : \Omega \rightarrow \mathbb{C}$ be analytic. Suppose that there is a convergent sequence $\{z_n\}_{n=1}^{\infty} \subset \Omega$, $z_n \neq z_m$ for all $n \neq m$, and $\lim_{n \rightarrow \infty} z_n = z \in \Omega$, such that $f(z_n) = f(z)$ for all n . Show that f is a constant function.

(c) Let $f : \{z \mid |z| \leq 1\} \rightarrow \mathbb{C}$ be continuous. Suppose that f is analytic in the open unit disc $\{z \mid |z| < 1\}$, and f takes real values on the unit circle $\{z \mid |z| = 1\}$. Prove that f must be a constant function.

Part B. Real Analysis

1. (a) Suppose that $\{E_i\}_{i=1}^{\infty}$ is a disjoint sequence of measurable sets of a measure space (X, μ) and $\mu(E_i) > 0$ for all $i = 1, 2, 3, \dots$. Let $E = \bigcup_{i=1}^{\infty} E_i$. Is it true that $\mu(E) = \infty$? Give either a proof or a counterexample.

(b) Suppose that X contains an uncountable disjoint collection $\{F_\alpha\}_{\alpha \in A}$ of measurable sets and $\mu(F_\alpha) > 0$ for all $\alpha \in A$. Prove that there exists an $\varepsilon > 0$ such that the subcollection

$$\{F_\alpha \mid \alpha \in A, \mu(F_\alpha) > \varepsilon\}$$

is also uncountable.

2. Let f be a continuous function on $[0, 1]$. Find $\lim_{n \rightarrow \infty} \int_0^1 nt^{3n} f(t) dt$. Prove your assertion.

3. Suppose that f is a Lebesgue integrable function on the real line and g is a C^∞ function of compact support. Let

$$h(x) = \int_{-\infty}^{\infty} g(y)f(x-y) dy .$$

(a) Show h is bounded.

(b) Show h is differentiable.

(c) Show h is a C^∞ function.

(d) Use a well-known double integral theorem to show that h is Lebesgue integrable.

4. Let f, f_1, f_2, f_3, \dots , be functions defined on $[0, 1]$.

a. Give the precise definitions for the following statements.

(1) f_n converges to f uniformly on $[0, 1]$.

(2) f_n converges to f pointwise on $[0, 1]$.

(3) f_n converges to f pointwise almost everywhere on $[0, 1]$.

(4) f_n converges to f in measure on $[0, 1]$.

b. Construct an example of continuous real-valued functions $\{f_n\}_{n=1}^{\infty}$ on $[0, 1]$ converging pointwise to a limit function f but not converging uniformly.

c. Construct an example of nonnegative integrable functions $\{f_n\}_{n=1}^{\infty}$ on $[0, 1]$ converging pointwise almost everywhere to a limit integrable function f such that

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx \text{ exists but is not equal to } \int_0^1 f(x) dx.$$

d. Construct an example of measurable real-valued functions $\{f_n\}_{n=1}^{\infty}$ on $[0, 1]$ converging to 0 in measure but not converging to 0 pointwise everywhere.

Part C. Functional Analysis

1. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $\{a_n\}$ is a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n b_n < \infty$ whenever $b_n \geq 0$ and $\sum_{n=1}^{\infty} b_n^q < \infty$. Prove that $\sum_{n=1}^{\infty} a_n^p < \infty$.

2. (a) Let F be a proper closed subspace of a Banach space E . For $x \in E$, let $d(x, F) = \inf\{\|x - y\| : y \in F\}$ be the distance from x to F . Show that

$$\sup\{d(x, F) : x \in E, \|x\| \leq 1\} = 1.$$

(b) Let E be the Banach space with the sup norm of all continuous functions f on $[0, 1]$ with $f(0) = 0$. Let $F = \{f \in E : \int_0^1 f(t) dt = 0\}$. Prove that there is no g in the unit ball of E with $d(g, F) = 1$. [Hint: Suppose such a g exists, use the Hahn-Banach theorem to show that $|\int_0^1 g(t) dt| = 1$, and hence a contradiction.]