FILE

Analysis Exam

Department of Mathematics

Tulane University

September 2, 1993

This exam consists of two parts. Both parts consist of 8 problems. I want you to solve a total of 8 problems, 4 from each part. Express your results with precision and clarity. If you use a "well-known" theorem please describe it briefly.

Part 1

1) Evaluate the integral

$$\int_\gamma \frac{e^{iz}}{z} dz$$

where γ is the curve consisting of the semicircles $|z| = R, Im(z) > 0$ and $|z| = r(< R), Im(z) > 0$, together with the segments on the real axis $r \leq |z| \leq R$. Let $R \to \infty$ and $r \to 0$, and find the value of

$$\int_0^\infty \frac{sin x}{x} dx$$

2) Describe the entire functions f that satisfy the growth estimate

$$|f(z)| \leq M(2 + |z|)^{4/3} \quad \text{for all } z \in \mathbf{C}$$

for some constant M.

3) Determine the radius of convergence of the series

$$\sum_{n=0}^{\infty}(n!)^k \cdot [(kn)!]^{-1} z^n$$

where k is a fixed natural number.

4) Assume that f is holomorphic (analytic) in the unit disk D and satisfies

$$f''(3^{-n}) = -f(3^{-n})$$

for $n = 1, 2, 3, \cdots$. Prove that f admits an extension to the whole complex plane and that the extension is an entire function.

5) Compute

$$\int_0^\infty sin(\pi x)[x(1-x^2)]^{-1} dx.$$

6) Discuss the mapping properties of the function

$$q(z) = \frac{1}{2}(z + z^{-1})$$

7) Determine the number of zeros of $9z^5 + 5z - 3$ in the annulus $\{z \in \mathbf{C}: \frac{1}{2} < |z| < 5\}$.

8) Characterize the entire functions that have a pole at ∞. Use your result to give a description of meromorphic functions that have a pole at ∞.

Part 2

1) Let $\alpha,\ \beta > 0$. Define the function $f : [0,1] \rightarrow \mathbf{R}$ via

$$f(x) = x^\alpha sin(x^{-\beta}) \text{ for } x \neq 0 \text{ and } f(0) = 0.$$

Describe the complete range of α, β for which f is of bounded variation.

2) Let f be a continuous function on $[0,1]$. Find

$$\lim_{n\to\infty} \int_0^1 n[e^{t^{-3n}} - 1]f(t)dt$$

Prove your assertion.

3) Let E be the Banach space of all continuous functions defined on $[0,1]$ that satisfy $f(0) = 0$. The norm on E is the supremum over the interval. Let F be the set

$$\{f \in E : \int_0^1 f(t)dt = 0\}$$

a) Check that F is a subspace.

b) Describe the set $F^{\perp}$, with respect to the dual of E. What is the dimension of $F^{\perp}$. **Hint.** Think of F as the kernel of a functional.

c) Is F closed?.

4) Consider the set

$$A_n = \{y \in \mathbf{R} : y = x(mod\ 1) \text{ for } x \in \mathbf{R} \text{ that satisfies } \sum_{j=1}^{n} j^{-1} \leq x \leq \sum_{j=1}^{n+1} j^{-1}\}$$

Here $x(mod\ 1)$ is the decimal part of x, therefore $y = x(mod\ 1) \in [0,1)$. Let f_n be the characteristic function of the set A_n.

a) Does $f_n \to 0$ almost everywhere?.

b) Does $f_n \to 0$ in measure?.

c) Does $f_n \to 0$ in L^p for some $p > 1$?.

5) Let

$$f_n(x) = \int_0^\infty (1 + y/n)^n e^{-xy} dy.$$

Prove that for $x > 1$, f_n has a limit as $n \to \infty$. Discuss the nature of the convergence.

6) Let $X = L^1[0,1]$. Define the **convolution** of f and g as

$$h(x) = f * g(x) = \int_0^x f(t)g(x-t)dt.$$

a) Show that this operation is well defined in X.

b) Show that if $f \in X$ and $g \in C^\infty$, then $h \in C^\infty$ and compute the derivatives of h.

7) Let f be an integrable function. Prove that $f = 0$ almost everywhere if and only if $\int_E f = 0$ for any measurable set E.

8) Let $f : [0,1] \to \mathbf{R}$ be Lebesgue integrable. Assume that f is differentiable at $x = 0$ and $f(0) = 0$. Show that $g : [0,1] \to \mathbf{R}$ defined by $g(x) = f(x)x^{-3/2}$ for $x \in (0,1]$ and $g(0) = 0$ is Lebesgue integrable.